MANN-HALPERN ITERATION METHODS IMPROVE HILBERT SPACES

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Abstract. In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming, the Mann's iterative method and the Halpern's method for finding a fixed point of a nonexpansive mapping and a common fixed point of a nonexpansive Hilbert spaces.
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1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle .,. \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote by $P_C(x)$ the metric projection from $x \in H$ onto C. Let T be a nonexpansive mapping on C, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

For finding a fixed point of a nonexpansive mapping T on C, in 1953, Mann [3] proposed the following method:

$$\begin{aligned} &x_0 \in C \quad \text{any element,} \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \end{aligned} \tag{1.1}$$

that converges only weakly, in general (see [4] for an example). In 1967, Halpern [5] firstly proposed the following iteration process:

$$x_{n+1} = \beta_n u + (1-\beta_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where u, x_0 are two fixed elements in C and $\{\beta_n\} \subset (0,1)$. He pointed out that the conditions $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$ are necessary in the sense that, if the iteration (1.2) converges to a fixed point of T, then these conditions must be satisfied. Further, the iteration method was investigated by Lions [6], Reich [7], Wittmann [8] and Song [9]. Recently, Alber [10] proposed the following descent-like method

$$x_{n+1} = P_C(x_n - \mu_n[x_n - Tx_n]), n \geq 0, \tag{1.3}$$

and proved that if $\{\mu_n\} : \mu_n > 0, \mu_n \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;

(ii) all weak accumulation points of $\{x_n\}$ belong to $F(T)$; and

(iii) $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_n\}$ converges weakly to $\tilde{x}$.

This problem is solved very recently in [14] and [15]. In this works, C_n and Q_n are replaced by two halfspaces and y_n is the right hand side of (1.3) with a modification.

In this paper, using the idea, we introduce the following new iteration processes:

$$\begin{aligned} x_0 &\in H \quad \text{any element,} \\ z_n &= \alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n), \\ y_n &= \beta_n x_0 + (1-\beta_n) P_C T z_n, \\ H_n &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \ge 0; \end{aligned} \tag{1.4}$$

for a nonexpansive mapping $T : C \to H$ and a nonexpansive semigroup $\{T(t) : t > 0\}$ on C, respectively. We shall prove the strong convergence of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.4) to a fixed point of T and a common fixed point of the nonexpansive semigroup $\{T(t) : t > 0\}$, respectively.

Later, the symbols $\rightharpoonup$ and $\to$ denote weak and strong convergences, respectively.

2. Strong convergence to a fixed point of nonexpansive mappings

We formulate the following facts needed in the proof of our results.

Lemma 2.1 [16]. *Let H be a real Hilbert space H. There holds the following identity:* $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$.

Lemma 2.2 [12]. *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z - x\| \le \|y - x\|$ for all $y \in C$, and $z = P_C(x)$ if and only if $\langle z - x, y - z\rangle \ge 0$ for all $y \in C$, where P_C is the metric projection of H on C.*

Lemma 2.3. (Demiclosedness principle) [17]. *If C is a nonempty closed convex subset of a real Hilbert space H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

Lemma 2.4 [17]. *Every Hilbert space H has Randon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there hodls $x_n \to x$.*

Now, we are in a position to prove the following result.

Theorem 2.5. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \to 1$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.4) converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.*

Proof. First, note that

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)$$

is equivalent to

$$\langle (1-\beta_n)x_n + \beta_n x_0 - y_n, z\rangle \le \langle x_n - y_n, x_n\rangle - \frac{1}{2}\|y_n - x_n\|^2 + \frac{\beta_n}{2}\|x_0\|^2.$$

Thus, H_n is a halfspace. It is clear that $F(T_1) = F(T_1P_C) := \{p \in H : T_1P_C(p) = p\}$ for any mapping T_1 from C into C. Taking $T_1 = P_CT$ and using Lemma 2.6 in [15] with $S = P_CT$, we have that $F(T) = F(P_CTP_C)$. Hence, by the convexity of $\|.\|^2$ and the nonexpansive property of P_C, we obtain for any $p \in F(T)$ that $p = P_CTP_C(p)$, and hence

$$\begin{aligned}\|z_n - p\|^2 &= \|\alpha_n P_C(x_n) - p + (1-\alpha_n)P_CTP_C(x_n)\|^2\\ &= \|\alpha_n(P_C(x_n) - P_C(p)) + (1-\alpha_n)[P_CTP_C(x_n) - P_CTP_C(p)]\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|P_C(x_n) - P_C(p)\|^2\\ &\le \|x_n - p\|^2.\end{aligned}$$

By the similar argument and Lemma 2.1 with $x = x_0 - p$ and $y = x_n - p$, we also obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n x_0 + (1-\beta_n)P_CTz_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|P_CTz_n - P_CTp\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|z_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|x_n - p\|^2\\ &= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2)\\ &= \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle).\end{aligned}$$

Therefore, $p \in H_n$ for all $n \ge 0$. It means that $F(T) \subset H_n$ for all $n \ge 0$.

Next, we show by mathematical induction that $F(T) \subset H_n \cap W_n$ for each $n \ge 0$. For $n = 0$, we have $W_0 = H$, and hence $F(T) \subset H_0 \cap W_0$. Suppose that x_i is given and $F(T) \subset H_i \cap W_i$ for some $i > 0$. There exists a unique element $x_{i+1} \in H_i \cap W_i$ such that $x_{i+1} = P_{H_i \cap W_i}(x_0)$. Therefore, by Lemma 2.2,

$$\langle x_{i+1} - x_0, p - x_{i+1}\rangle \ge 0$$

for each $p \in H_i \cap W_i$. Since $F(T) \subset H_i \cap W_i$, we get $F(T) \subset W_{i+1}$. So, we have $F(T) \subset H_{i+1} \cap W_{i+1}$.

Further, since $F(T)$ is a nonempty closed convex subset of H, there exists a unique element $u_0 \in F(T)$ such that $u_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{H_n \cap W_n}(x_0)$, we obtain

$$\|x_{n+1} - x_0\| \le \|z - x_0\|$$

for every $z \in H_n \cap W_n$. As $u_0 \in F(T) \subset W_n$, we get

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\| \quad n \ge 0. \tag{2.1}$$

This implies that $\{x_n\}$ is bounded. So, $\{P_CTP_C(x_n)\}, \{z_n\}$ and $\{Tz_n\}$ are also bounded.

Now, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.2}$$

From the definition of W_n and Lemma 2.2, it follows that $x_n = P_{W_n}(x_0)$. As $x_{n+1} \in H_n \cap W_n$, we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\| \quad n \geq 0.$$

Therefore, $\{\|x_n - x_0\|\}$ is a nondecreasing and bounded sequence. So, there exists $\lim_{n\to\infty} \|x_n - x_0\| = c$. On the other hand, from $x_{n+1} \in W_n$, we have $\langle x_n - x_0, x_{n+1} - x_n\rangle \geq 0$ and hence

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 - (x_{n+1} - x_0)\|^2\\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+1} - x_0\rangle + \|x_{n+1} - x_0\|^2\\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \quad \forall n \geq 0.\end{aligned}$$

Thus, (2.2) is followed from the last inequality and $\lim_{n\to\infty} \|x_n - x_0\| = c$.

Since $\alpha_n \to 1$ and $\{x_n\}, \{P_C T P_C(x_n)\}$ are bounded, we have from (1.4) that

$$\lim_{n\to\infty} \|z_n - P_C(x_n)\| = \lim_{n\to\infty} (1 - \alpha_n)\|P_C(x_n) - P_C T P_C(x_n)\| = 0. \tag{2.3}$$

On the other hand, since $x_{n+1} \in H_n$ we have that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \beta_n(\|x_0\| + 2\langle x_n - x_0, x_{n+1}\rangle)\}.$$

Therefore, from (2.2), the boundedness of $\{x_n\}$, $\beta_n \to 0$ and the last inequality, it follows that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{2.4}$$

This together with (2.2) implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.5}$$

Noticing that $P_C T z_n = y_n - \beta_n(x_n - P_C T z_n) + \beta_n(x_n - x_0)$, we have

$$\|x_n - P_C T z_n\| \leq \|x_n - y_n\| + \beta_n\|x_n - P_C T z_n\| + \beta_n\|x_n - x_0\|.$$

From (2.1) and the last inequality, it follows that

$$\|x_n - P_C T z_n\| \leq \frac{1}{1 - \beta_n}\Big(\|x_n - y_n\| + \beta_n\|u_0 - x_0\|\Big).$$

By $\beta_n \to 0$ ($\beta_n \leq 1 - \beta$ for some $\beta \in (0, 1)$), (2.5) and the last inequality, we obtain

$$\lim_{n\to\infty} \|x_n - P_C T z_n\| = 0. \tag{2.6}$$

Further, we have that $P_C T z_n = P_C P_C T z_n$, and hence

$$\begin{aligned}\|z_n - P_C T z_n\| &\leq \|z_n - P_C(x_n)\| + \|P_C(x_n) - P_C P_C(T z_n)\|\\ &\leq \|z_n - P_C(x_n)\| + \|x_n - P_C T z_n\|.\end{aligned}$$

So, from (2.3), (2.6) and the last inequality, it follows that

$$\lim_{n\to\infty} \|z_n - P_C T z_n\| = 0. \tag{2.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that convegers weakly to some element $p \in H$ as $j \to \infty$. From (2.6) and (2.7), we also have that $\{z_{n_j}\}$ converges weakly to p. Since $\{z_n\} \subset C$, we obtain that $p \in C$. By Lemmas 2.3 and (2.7), $p \in F(P_C T) = F(T)$ by Lemma 2.6 in [15] with S replaced by T.

Now, from (2.1) and the weakly lower semicontinuity of the norm it implies that

$$\|x_0 - u_0\| \leq \|x_0 - p\| \leq \lim\inf_{j\to\infty} \|x_0 - x_{n_j}\| \leq \lim\sup_{j\to\infty} \|x_0 - x_{n_j}\| \leq \|x_0 - u_0\|.$$

Thus, we obtain $\lim_{j\to\infty} \|x_0 - x_{n_j}\| = \|x_0 - u_0\| = \|x_0 - p\|$. This implies $x_{k_j} \to p = u_0$ by Lemma 2.4. By the uniqueness of the projection $u_0 = P_{F(T)}(x_0)$, we have that $x_n \to u_0$. From (2.5) and (2.6)-(2.7), we also get $y_n \to u_0$ and $z_n \to u_0$, respectively. This completes the proof.

Corollary 2.6. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in [0,1] such that such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ y_n &= \beta_n x_0 + (1-\beta_n) P_C T P_C(x_n), \\ H_n &= \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

Proof. By putting $\alpha_n \equiv 1$ in Theorem 2.5, we obtain the conclusion.

Corollary 2.7. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in [0,1] such that $\alpha_n \to 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ y_n &= P_C T(\alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n)), \\ H_n &= \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

Proof. By putting $\beta_n \equiv 0$ in Theorem 2.5, we obtain the conclusion.

Phương pháp lặp Mann - Halpern cải biên trong không gian Hilbert

Tóm tắt

Tron bài báo này, chúng tôi giới thiệu một số phương pháp lăp mới dựa trên phương

pháp lai ghép trong qui hoạch toán học, dựa trên phương pháp lặp của Mann và Halpern để tìm một điểm bất động và một điểm chung của không gian Hilbert.

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