ON A QUASI UNMIXED ARTINIAN MODULE

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, A an Artinian R-module and M a finitely generated R-module. Consider the following property:

$$\operatorname{Ann}_R(0:_A \mathfrak{p}) = \mathfrak{p} \text{ for all prime ideals } \mathfrak{p} \supseteq \operatorname{Ann}_R A. \qquad (*)$$

We say that A is *quasi unmixed* if $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ for all $\widehat{\mathfrak{p}} \in \min \operatorname{Att}_{\widehat{R}} A$. In [14] the author and L. T. Nhan showed that if a quasi unmixed Artinian module A satisfies property $(*)$ then the ring $R/\operatorname{Ann}_R A$ is catenary and $\dim(R/\operatorname{Ann}_R A) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$. In this paper we give an example to show that the conversion of this result is not true in general. In [14] we also had that for an integer $i \geq 0$, if the local cohomology module $H^i_{\mathfrak{m}}(M)$ is quasi unmixed then $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) if and only if the ring $R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ is catenary and $\dim(R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}}(H^i_{\mathfrak{m}}(M)))$. Also by above example we will show that this result is not true for local cohomology with arbitrary support.

Key words: *Quasi unmixed Artinian modules, local cohomology modules, Noetherian dimension, catenary rings, power series rings.*

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, A an Artinian R-module, and M be a finitely generated R-module. For each ideal I of R, we denote by $\operatorname{Var}(I)$ the set of all prime ideals containing I.

It is clear that $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R M)$. Therefore it is natural to ask the dual property for Artinian modules:

$$\operatorname{Ann}_R(0:_A \mathfrak{p}) = \mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R A). \qquad (*)$$

If R is complete with respect to $\mathfrak{m}$-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian R-modules. However, there are Artinian modules which do not satisfy this property. For example, by [4, Example 4.4], the Artinian R-module $H^1_{\mathfrak{m}}(R)$ does not satisfy the property (*), where R is the Noetherian local domain of dimension 2 constructed by M. Ferrand and D. Raynaud [7] (see also [12, App. Ex. 2]) such that its $\mathfrak{m}$-adic completion $\widehat{R}$ has an associated prime $\mathfrak{q}$ of dimension 1.

Note that if R is complete with respect to $\mathfrak{m}$-adic topology then the property (*) is satisfied for all Artinian R-modules. In this case, the Matlis duality is useful to study the relation between the category of Artinian R-modules and the category of Noetherian R-modules. In case the ring R is not complete, it seems to us that the study of the property (*) for Artinian modules is very important since it gives a lot of information on the base ring R, see [3], [13], [14], [16], ... The main theorem of [3] states that if M is a finitely generated R-module with $\dim M = d$ then the top local cohomology module $H^d_{\mathfrak{m}}(M)$ satisfies the property (*) if and only if the ring $R/\operatorname{Ann}_R(H^d_{\mathfrak{m}}(M))$ is catenary. Note that $\operatorname{Att}_R H^d_{\mathfrak{m}}(M) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} = d\}$. In [14], Nhan and the author defined unmixed Artinian modules.

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Definition 1.1. An Artinian R-module A is said to be *equidimensional* if

$$\dim(R/\mathfrak{p}) = \dim(R/\operatorname{Ann}_R A)$$

for all attached primes $\mathfrak{p} \in \min \operatorname{Att}_R A$. We say that A is *quasi unmixed* if the $\widehat{R}$-module A is equidimensional, i.e. $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ for all $\widehat{\mathfrak{p}} \in \min \operatorname{Att}_{\widehat{R}} A$. If $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ for all $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}} A$ then A is called *unmixed.*

By above definition, $H^d_{\mathfrak{m}}(M)$ is unmixed. So the following main result in [14] generalized the main result of [3].

Theorem 1.2. [14, Theorem 1.1, 1.2] *(a) Assume that A is quasi unmixed. If A satisfies the property (*) then the ring $R/\operatorname{Ann}_R A$ is catenary and* $\dim(R/\operatorname{Ann}_R A) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$.

(b) Assume that $H^i_{\mathfrak{m}}(M)$ is quasi-unmixed. Then the following statements are equivalent:

(i) $H^i_{\mathfrak{m}}(M)$ satisfies the property (*).

(ii) $\dim\left(R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))\right) = \dim\left(\widehat{R}/\operatorname{Ann}_{\widehat{R}} H^i_{\mathfrak{m}}(M)\right)$ *and the ring* $R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ *is catenary.*

It is natural to consider the conversion of Theorem 1.2 (a). In this paper we give an example to show that the conversion of this result is not true in general. This example also shows that the result in Theorem 1.2 (b) is not true for local cohomology with arbitrary support. That is the main purpose of this paper.

This paper is divided into 2 sections. In the next section we will introduce the example after some preliminaries.

2. An quasi unmixed Artinian modules

Firstly, we remind some results on attached primes and Noetherian dimension of Artinian modules. The theory of secondary representation was introduced by I. G. Macdonald [11] which is in some sense dual to that of primary decomposition for Noetherian modules. Note that every Artinian R-module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called *the set of attached prime ideals* of A, and denoted by $\operatorname{Att}_R A$.

Lemma 2.1. [11] *The set of all minimal elements of* $\operatorname{Att}_R A$ *is exactly the set of all minimal elements of* $\operatorname{Var}(\operatorname{Ann}_R A)$. *In particular,*

$$\dim(R/\operatorname{Ann}_R A) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Att}_R A\}.$$

R. N. Roberts [15] introduced the concept of Krull dimension for Artinian modules. D. Kirby [10] changed the terminology of Roberts to Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules. Here we use the terminology of Kirby [10]. Denote by $\text{N-dim}_R A$ the Noetherian dimension of A.

Note that A has a natural structure as an $\widehat{R}$-module. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule of A. Therefore A is an Artinian $\widehat{R}$-module and $\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A$. So, without any confusion, we can write $\text{N-dim}\, A$ instead of $\text{N-dim}_R A$ or $\text{N-dim}_{\widehat{R}} A$.

Lemma 2.2. [2, 8.2.4 and 8.2.5] $\mathrm{Att}_R A = \{\widehat{\mathfrak{p}} \cap R \mid \widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{R}} A\}$.

The following properties on dimension of Artinian modules have been proved in [4].

Lemma 2.3. *The following statements hold.*
(i) $\dim(R/\mathrm{Ann}_R A) \geq \mathrm{N\text{-}dim}_R A$.
(ii) If A satisfies the property (*) then $\dim(R/\mathrm{Ann}_R A) = \mathrm{N\text{-}dim}_R A$.

Now, our example is contructed base on the domain given by C. Huneke and A. Taylor [9]. Let I be an ideal of R. Remind that an R-module N is said to be *I-cofinite* if $\mathrm{Supp}(N) \subseteq \mathrm{Var}(I)$ and $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for all $i \geq 0$. Using Matlis duality we have that an R-module is $\mathfrak{m}$-cofinite if and only if it is an Artinian module. As a consequence, the local cohomology module $H^i_{\mathfrak{m}}(M)$ is $\mathfrak{m}$-cofinite for any finitely generated R-module M. In [8], R. Hartshorne posed the question that whether $H^i_I(M)$ is I-cofinite for all i? However, R. Hartshorne gave an example to show that this result is not true in general. He also proved that if R is complete regular local ring, $\mathfrak{p}$ is dimension one prime ideal of R then $H^i_{\mathfrak{p}}(M)$ is finitely generated for all i. D. Delfino and T. Marley [5] generalized Hartshorne's result for any commutative Noetherian local ring and $\mathfrak{p}$ is dimension one prime ideal of R. They also consider the top local cohomology module.

Lemma 2.4. [5, Theorem 3] *Let I be an ideal of R and M a finitely generated R-module of dimension d. Then $H^d_I(M)$ is I-cofinite.*

Theorem 2.5. *Let k be a field of characteristic 0, $R = \left(\frac{k[x,y,u,v]}{(f)}\right)_{\mathfrak{m}}$, where $k[x, y, u, v]$ is a polynomial ring, $f = xy - ux^2 - vy^2$ and $\mathfrak{m} = (x, y, u, v)$. Set $\mathfrak{p} = (y, u, v)R$. Then*
(i) R is three-dimension catenary domain, $\mathfrak{p}$ is prime ideal and $\mathrm{ht}(\mathfrak{p}) = 2$ *(* [9, Example 6.2]*).*
(ii) $0 \neq H^3_{\mathfrak{p}}(R)$ is quasi unmixed Artinian module,

$$\dim R/\mathrm{Ann}_R H^3_{\mathfrak{p}}(R) = \dim \widehat{R}/\mathrm{Ann}_{\widehat{R}} H^3_{\mathfrak{p}}(R) = 3$$

but $H^3_{\mathfrak{p}}(R)$ does not satisfies the property (*).

First, we prove the following result.

Lemma 2.6. *In the power series ring $k[[x, y, u, v]]$ we have the expression*

$$f = xy - ux^2 - vy^2 = (x - vy + a_3 + a_4 + \cdots)(y - ux + b_3 + b_4 + \cdots),$$

where a_i, b_i, for all $i \geq 3$, is homogeneous of degree i and lies in $(y, u, v)k[[x, y, u, v]]$.

Proof. We prove the existence of a_i, b_i, for all $i \geq 3$ by induction on i. We also show that every term a_i or b_i in each factor of f has positive degree in x or in y. We have

$$xy - ux^2 - vy^2 = xy - ux^2 + b_3x - vy^2 - xyuv + a_3y + \text{term of degree} \geq 5.$$

Compare the degree 4 in both sides implies $xb_3 + ya_3 + xyuv = 0$. Set $a_3 = -1/2xuv$ and $b_3 = -1/2yuv$. Then a_3, b_3 satisfies the requirement. Assume that we choosed a_i, b_i, where $i \leq n - 1$ and $n \geq 4$. Comparing the degree $n + 1$, we have

$$xb_n - yvb_{n-1} + a_3b_{n-2} + \cdots + ya_n - xua_{n-1} + b_3a_{n-2} = 0.$$

Then

$$x(b_n - ua_{n-1}) + y(a_n - vb_{n-1}) + \sum_{i+j=n+1;i,j\geq 3} a_i b_j = 0.$$

By induction, $a_i, b_j \in (y, u, v)$ and a_i, b_j always contain one of the following monomial x^2, xy or y^2, where $i + j = n + 1, i, j \geq 3$. Hence

$$x(b_n - ua_{n-1} + \sum a_i' b_j') + y(a_n - vb_{n-1} + \sum a_i'' b_j'') = 0,$$

where the degree in x or in y of $a_i' b_j'$ is positive and the degree in y of $a_i'' b_j''$ is positive. Choose

$$a_n = vb_{n-1} - \sum a_i'' b_j''; \; b_n = ua_{n-1} - \sum a_i' b_j',$$

we have the requirement. □

Proof of Theorem 2.5. Let $\widehat{R}$ be the completion of R with respect to $\mathfrak{m}$-adic topology. We can prove that $x - vy + a_3 + a_4 + \cdots$ and $y - uv + b_3 + b_4 + \cdots$ in Lemma 2.6 are irreducible. Set $\widehat{\mathfrak{p}}_1 = (x - vy + a_3 + a_4 + \cdots)\widehat{R}$ and $\widehat{\mathfrak{p}}_2 = (y - uv + b_3 + b_4 + \cdots)\widehat{R}$. They are prime ideals and

$$\dim \widehat{R}/\widehat{\mathfrak{p}}_1 = \dim \widehat{R}/\widehat{\mathfrak{p}}_2 = 3.$$

On the other hand by Lemma 2.6, we have $(f)\widehat{R} = \widehat{\mathfrak{p}}_1 \cap \widehat{\mathfrak{p}}_2$. Hence $\operatorname{Assh} \widehat{R} = \{\widehat{\mathfrak{p}}_1, \widehat{\mathfrak{p}}_2\}$. Since $\mathfrak{p}\widehat{R} + \widehat{\mathfrak{p}}_1 = \mathfrak{m}\widehat{R}$, by Lichtenbaum-Hartshorne Vanishing Theorem [2, Theorem 8.2.1], $H^3_{\mathfrak{p}}(R) \neq 0$. Further more

$$\mathfrak{p}\widehat{R} + \widehat{\mathfrak{p}}_2 = \mathfrak{p}\widehat{R} \subsetneq \mathfrak{m}\widehat{R}.$$

Hence by [6, Corollary 2], we have $\operatorname{Att}_{\widehat{R}} H^3_{\mathfrak{p}}(R) = \{\widehat{\mathfrak{p}}_1\}$. Then $\widehat{\mathfrak{p}}_1 \cap R \in \operatorname{Att}_R H^3_{\mathfrak{p}}(R)$. Since $0 \subseteq \widehat{\mathfrak{p}}_1 \cap R$ are prime ideals of R and

$$\dim R/(\widehat{\mathfrak{p}}_1 \cap R) \geq \dim \widehat{R}/\widehat{\mathfrak{p}}_1 = 3,$$

$0 = \widehat{\mathfrak{p}}_1 \cap R \in \operatorname{Att}_R H^3_{\mathfrak{p}}(R)$. This implies $\operatorname{Ann}_R H^3_{\mathfrak{p}}(R) = 0$. So $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R H^3_{\mathfrak{p}}(R))$. Because $H^3_{\mathfrak{p}}(R)$ is Artinian module then $0 :_{H^3_{\mathfrak{p}}(R)} \mathfrak{p} = \operatorname{Hom}(R/\mathfrak{p}; H^3_{\mathfrak{p}}(R))$ has finite length. This prove that $\operatorname{Ann}_R(0 :_{H^3_{\mathfrak{p}}(R)} \mathfrak{p}) \neq \mathfrak{p}$, or $H^3_{\mathfrak{p}}(R)$ does not satisfies the property $(*)$. □

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TÓM TẮT
VỀ MỘT MÔĐUN ARTIN TỰA KHÔNG TRỘN LẪN

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Cho $(R, \mathfrak{m})$ là một vành địa phương Noether, A là một R-môđun Artin và M là một R-môđun hữu hạn sinh. Xét tính chất sau

$$\operatorname{Ann}_R(0:_A \mathfrak{p}) = \mathfrak{p} \text{ với mọi iđêan nguyên tố } \mathfrak{p} \supseteq \operatorname{Ann}_R A. \qquad (*)$$

Ta nói rằng A là *tựa không trộn lẫn* nếu $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ với mọi $\widehat{\mathfrak{p}} \in \min \operatorname{Att}_{\widehat{R}} A$. Trong [14] tác giả và L. T. Nhan đã chỉ ra rằng nếu một môđun Artin tựa không trộn lẫn A thỏa mãn tính chất $(*)$ thì vành $R/\operatorname{Ann}_R A$ là catenary và $\dim(R/\operatorname{Ann}_R A) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$. Trong bài báo này chúng tôi đưa ra ví dụ chứng tỏ rằng điều ngược lại của kết quả trên nhìn chung không đúng. Cũng trong [14] các tác giả đã chứng minh với mỗi $i \geq 0$, nếu môđun đối đồng điều địa phương $H^i_{\mathfrak{m}}(M)$ là tựa không trộn lẫn thì $H^i_{\mathfrak{m}}(M)$ thỏa mãn tính chất (*) nếu và chỉ nếu vành $R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ là catenary và $\dim(R/\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}}(H^i_{\mathfrak{m}}(M)))$. Ví dụ trong bài báo này cũng chỉ ra kết quả trên không đúng cho các môđun đối đồng điều địa phương với giá bất kỳ.

Từ khóa: *Môđun Artin tựa không trộn lẫn, môđun đối đồng điều địa phương, Chiều Noether, vành catenary, vành chuỗi lũy thừa hình thức.*

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