SOME NEW NORMALITY CRITERIAS FOR FAMILY OF MEROMORPHIC FUNCTIONS CONCERNING THE RESULT OF PANG-ZALCMAN

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Abstract

The paper concerns interesting problems related to the field of Complex Analysis, in particular, Nevanlinna theory of meromorphic functions and applications. We prove some new normal criterias for family of meromorphic functions concerning the normality criteria due to Pang and Zalcman [2]. Our main result is stated: Let a be nonzero complex value and $n \geq 2$ be a positive integer, and let $n_1, \dots, n_{k-1}$ be nonnegative integers, n_k be positive integer ($k \geq 1$). Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D all of whose zeros have multiplicity at least k such that $E_f = \{z : f^n(z)(f')^{n_1}(z) \cdots (f^{(k)})^{n_k}(z) - a = 0\}$ has at most one point in D, for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family. In our best knowledge, it is a new result which is supplement the result of Pang-Zalcman in this trend.

Keywords: *entire function, meromorphic function, normal family, Nevanlinna theory, Zalcman's Lemma.*

1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_v\} \subset \mathcal{F}$, there exists a subsequence $\{f_{v_i}\}$ such that $\{f_{v_i}\}$ converges spherically locally uniformly in D, to a meromorphic function or ∞.

In 1999, Pang and Zalcman [2] proved the normality criteria as follows:

Theorem 1. *Let n and k be natural numbers and $\mathcal{F}$ be a family of holomorphic functions in a domain D all of whose zeros have multiplicity at least k. Assume that $f^n f^{(k)} - 1$ is non-vanishing for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

The main purpose of this paper is to establish some normality criterias for the case of meromorphic functions in above result. Namely, we prove

Theorem 2. *Let a be nonzero complex value and $n \geq 2$ be a positive integer, and let $n_1, \dots, n_{k-1}$ be nonnegative integers, n_k be positive integer ($k \geq 1$). Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D all of whose zeros have multiplicity at least k such that $E_f = \{z : f^n(z)(f')^{n_1}(z) \cdots (f^{(k)})^{n_k}(z) - a = 0\}$ has at most one point in D, for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

In Theorem 2, if $E_f = \varnothing$, this means $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} \neq a$, then we obtain the following result.

Corollary 1. *Let a be nonzero complex value and $n \geq 2$ be a positive integer, and let $n_1, \ldots, n_{k-1}$ be nonnegative integers, n_k be positive integer ($k \geq 1$). Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D all of whose zeros have multiplicity at least k such that $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} \neq a$ in D for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

In Corollary 1, taking $n_j = 0, j = 1, \ldots, k-1$, $n_k = 1$, we get

Corollary 2. *Let a be nonzero complex value and $k, n \geq 2$ are positive integer. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D all of whose zeros have multiplicity at least k such that $f^n f^{(k)} - a$ is non-vanishing for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

2 Lemma

In order to prove Theorem 2, we need the following lemmas:

Lemma 1 (Zalcman's Lemma, [4]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in the unit disc D satisfying all zeros of functions in $\mathcal{F}$ having multiplicity at least p, and all poles at least q. Let α be real number satisfying $-p < \alpha < q$. Then, $\mathcal{F}$ is not normal at z_0 if and only if there exist*
(i) a number $0 < r < 1$;
(ii) points z_n with $|z_n| < r$, $z_n \to z_0$;
(iii) functions $f_n \in \mathcal{F}$;
(iv) positive numbers $\rho_n \to 0^+$;
such that $g_n(\xi) = \rho_n^\alpha f_n(z_n + \rho_n \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros and poles have multiplicity at least p, q respectively. Moreover, g has order at most 2.

Lemma 2 ([1]). *Let g be a entire function and M is a positive constant. If $g^{\#}(\xi) \leqslant M$ for all $\xi \in \mathbb{C}$, then g has order at most one.*

Remark 3. In Lemma 1, if $\mathcal{F}$ is a family of holomorphic functions, then by Hurwitz theorem, g is a holomorphic function. Therefore, by Lemma 2, the order of g is not greater than 1.

Lemma 3. *Let f be a non-constant rational function on the complex plane, and let k be a positive integer. Assume that all of zeros of f have multiplicity at least k. We consider a nonzero complex number a and nonnegative integers $n, n_1, \ldots, n_k$ satisfying $n_1 + \cdots + n_k \geq 1$ and $n \geq 2$. Then the equation*

$$f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - a = 0$$

has at least two distinct roots.

Proof. We distinguish two cases.

Case 1. f is a polynomial. Since all zeros of f have multiplicity at least k, we get that the polynomial $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k} - a$ has degree at least 2. Suppose that $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k} - a$ has unique zero z_0. We have

$$\begin{aligned} &f^n(z)(f'(z))^{n_1} \cdots (f^{(k)}(z))^{n_k} - a \\ &= A(z - z_0)^\ell, \quad \ell \geq 2, A \neq 0. \end{aligned} \tag{2.1}$$

This implies,

$$\begin{aligned} &\left(f^n(z)(f'(z))^{n_1} \cdots (f^{(k)}(z))^{n_k}\right)' \\ &= A\ell(z - z_0)^{\ell - 1} \end{aligned}$$

Hence, the function $\left(f^n(f')^{n_1} \cdots (f^{(k)})^{n_k}\right)'$ has unique zero z_0. On the other hand, since $n \geq 2$, we have that a zero of f is also a zero of $\left(f^n(f')^{n_1} \cdots (f^{(k)})^{n_k}\right)'$. Hence, the polynomial f has unique zero z_0. This is impossible, by (2.1) and by the fact that $a \neq 0$.

Hence, the equation $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} - a = 0$ has at least two distinct roots, in this case.

Case 2. f is not a polynomial.

Since, f is a rational function with all zeros having multiplicity at least k, we can write

$$f = A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{d_1}(z-\beta_2)^{d_2}\cdots(z-\beta_t)^{d_t}}, \tag{2.2}$$

where the integers s, t can be zero, and $m_i \geq k$.

Set $M = m_1+\cdots+m_s, N = d_1+\cdots+d_t \geq t$, ($M = 0$, if $s = 0$, and $N = 0$ if $t = 0$). Then $M \geq sk$, and $N \geq t$.

We have

$$f^n(z) = A^n\frac{\prod_{k=1}^{s}(z-\alpha_k)^{nm_k}}{\prod_{l=1}^{t}(z-\beta l)^{nd_l}},$$

and

$$f^{(j)} = A\frac{\prod_{k=1}^{s}(z-\alpha_k)^{m_k-j}}{\prod_{l=1}^{t}(z-\beta_l)^{d_l+j}}g_j(z),$$

$$(j = 1,\ldots,k) \tag{2.3}$$

where $g_j(z)$ is a polynomial with $\deg g_j \leq j(s+t-1)$. Therefore,

$$f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k} = \frac{\prod_{i=1}^{s}(z-\alpha_i)^{(n+\sum_{j=1}^{k}n_j)m_i-\sum_{j=1}^{k}n_j j}}{\prod_{\ell=1}^{t}(z-\beta_\ell)^{(n+\sum_{j=1}^{k}n_j)d_\ell+\sum_{j=1}^{k}n_j j}}g(z) := \frac{P(z)}{Q(z)}, \tag{2.4}$$

where $g(z) = A^{n+n_1+\ldots+n_k}\prod_{j=1}^{k}g_j^{n_j}(z)$, $\deg g \leq (\sum_{j=1}^{k}jn_j)(s+t-1)$.

Hence

$$\left(f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k}\right)' = \frac{\prod_{i=1}^{s}(z-\alpha_1)^{(n+\sum_{j=1}^{k}n_j)m_i-\sum_{j=1}^{k}n_j j-1}}{\prod_{\ell=1}^{t}(z-\beta_1)^{(n+\sum_{j=1}^{k}n_j)d_\ell+\sum_{j=1}^{k}n_j j+1}}g_1(z), \tag{2.5}$$

where $\deg g_1 \leq (\sum_{j=1}^{k}jn_j+1)(s+t-1)$.

$*$) *Suppose that* $f^n(f')^{n_1}...(f^{(k)})^{n_k} - a = 0$ *has unique zero* z_0.

Then, we can write

$$f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k} = a + \frac{B(z-z_0)^l}{\prod_{\ell=1}^{t}(z-\beta_\ell)^{(n+\sum_{j=1}^{k}n_j)d_1+\sum_{j=1}^{k}n_j j}}, \tag{2.6}$$

where $B \neq 0$, $\alpha_i \neq z_0$.

Then

$$\left(f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k}\right)' = \frac{(z-z_0)^{l-1}g_2(z)}{\prod_{\ell=1}^{t}(z-\beta_\ell)^{(n+\sum_{j=1}^{k}n_j)d_\ell+\sum_{j=1}^{k}n_j j+1}}, \tag{2.7}$$

where $g_2(z) = B(l-(n+\sum_{j=1}^{k}n_j)N - (\sum_{j=1}^{k}jn_j)t)z^t + b_{t-1}z^{t-1} + \cdots + b_0$; and $b_0,\ldots,b_{t-1}$ are constants.

Subcase 1. $l \neq (n+\sum_{j=1}^{k}n_j)N + (\sum_{j=1}^{k}jn_j)t$.

Then, from (2.4) and (2.6), we have

$\deg P \geq \deg Q$. Hence,

$$(n+\sum_{j=1}^{k} n_j)M - (\sum_{j=1}^{k} jn_j)s + \deg g$$
$$\geq (n+\sum_{j=1}^{k} n_j)N + (\sum_{j=1}^{k} jn_j)t.$$

On the other hand,

$$\deg g \leq (\sum_{j=1}^{k} jn_j)(s+t-1).$$

Therefore,

$$(n+\sum_{j=1}^{k} n_j)M - (\sum_{j=1}^{k} jn_j)s$$
$$+ (\sum_{j=1}^{k} jn_j)(s+t-1)$$
$$\geq (n+\sum_{j=1}^{k} n_j)N + (\sum_{j=1}^{k} jn_j)t.$$

Then, $M > N$.

From (2.5) and (2.7), we have

$$\prod_{i=1}^{s}(z-\alpha_i)^{(n+\sum_{j=1}^{k} n_j)m_i - \sum_{j=1}^{k} n_j j - 1} g_1(z)$$
$$\equiv (z-z_0)^{l-1} g_2(z).$$

Therefore, since $z_0 \neq \alpha_i, i = 1, \ldots, s$ and $\deg g_2 = t$, we have

$$(n+\sum_{j=1}^{k} n_j)M - (\sum_{j=1}^{k} jn_j + 1)s \leq t.$$

Then

$$(n+\sum_{j=1}^{k} n_j)M \leq (\sum_{j=1}^{k} jn_j + 1)s + t$$
$$\leq (\sum_{j=1}^{k} jn_j + 1)\frac{M}{k} + N$$
$$< (\sum_{j=1}^{k} n_j)M + (1+\frac{1}{k})M$$
$$\leq (n+\sum_{j=1}^{k} n_j)M.$$

This is a contradiction.

Subcase 2. $l = (n+\sum_{j=1}^{k} n_j)N + (\sum_{j=1}^{k} jn_j)t.$

If $M > N$, then by an argument simliar to the last part of Subase 1, we get a contradiction. Hence, $M \leq N$.

Since $z_0 \neq \alpha_j$ $(j = 1, \ldots, s)$ therefore, by (2.5) and (2.7) we have

$$l - 1 \leq \deg g_1 \leq (\sum_{j=1}^{k} jn_j + 1)(s+t-1).$$

Therefore,

$$(n+\sum_{j=1}^{k} n_j)N = l - (\sum_{j=1}^{k} jn_j)t$$
$$\leq \deg g_1 + 1 - (\sum_{j=1}^{k} jn_j)t$$
$$\leq (\sum_{j=1}^{k} jn_j + 1)(s+t-1) + 1 - (\sum_{j=1}^{k} jn_j)t$$
$$< (\sum_{j=1}^{k} jn_j)s + s + t \leq \frac{M}{k}(\sum_{j=1}^{k} jn_j + 1) + N$$
$$\leq \frac{N}{k}(\sum_{j=1}^{k} jn_j + 1) + N \leq (n+\sum_{j=1}^{k} n_j)N.$$

This is a contradiction.

*) *Suppose that $f^n(f')^{n_1}\dots(f^{(k)})^{n_k}-a=0$ is nowhere vanishing.*
Then

$$f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k}=\frac{P(z)}{Q(z)}$$
$$=a+\frac{B}{\prod_{\ell=1}^{t}(z-\beta_\ell)^{(n+\sum_{j=1}^{k}n_j)d_\ell+\sum_{j=1}^{k}n_j j}},\tag{2.8}$$

where B is a non-zero constant.

Hence, $\deg P=\deg Q$. Therefore, by (2.4)

$$M(n+\sum_{j=1}^{k}n_j)-s\sum_{j=1}^{k}jn_j+\deg g$$
$$=N(n+\sum_{j=1}^{k}n_j)+t\sum_{j=1}^{k}jn_j$$

On the other hand,

$$\deg g\le(\sum_{j=1}^{k}jn_j)(s+t-1).$$

Therefore,

$$M\ge N+\frac{\sum_{j=1}^{k}jn_j}{n+\sum_{j=1}^{k}n_j}.$$

Then, $M>N$.
By (2.8), we have

$$(f^n(z)(f'(z))^{n_1}\cdots(f^{(k)}(z))^{n_k})'$$
$$=\frac{-Bg_3(z)}{\prod_{\ell=1}^{t}(z-\beta_\ell)^{(n+\sum_{j=1}^{k}n_j)d_\ell+\sum_{j=1}^{k}jn_j+1}},\tag{2.9}$$

where

$$g_3(z)=((n+\sum_{j=1}^{k}n_j)N+t\sum_{j=1}^{k}jn_j)z^{t-1}$$
$$b_{t-2}z^{t-2}+\cdots+b_1,$$

$\deg g_3=t-1$, and $b_i\in\mathbb{C}$.

From (2.5) and (2.9), we have

$$(n+\sum_{j=1}^{k}n_j)M-(1+\sum_{j=1}^{k}jn_j)s$$
$$\le\deg g_3=t-1.$$

Hence

$$(n+\sum_{j=1}^{k}n_j)M\le(1+\sum_{j=1}^{k}jn_j)s+t-1$$
$$\le(1+\sum_{j=1}^{k}jn_j)\frac{M}{k}+N-1$$
$$<(\sum_{j=1}^{k}n_j+2)M-1<(\sum_{j=1}^{k}n_j+n)M.$$

This is a contradiction. We have completed the proof of Lemma 3. □

Lemma 4 ([3]). *Let f be a transcendental meromorphic function on the complex plane, and $a\neq0$ be a complex number. Assume that $n\ge2,n_1,\dots,n_k$ are nonnegative integers such that $n_1+\dots+n_k\ge1$. Then the equation*

$$f^n(f')^{n_1}\dots(f^{(k)})^{n_k}-a=0$$

has infinitely roots.

3 Proof of Theorem 2

Proof of Theorem 2. Without loss the generality, we may asssume that D is the unit disc. Suppose that $\mathcal{F}$ is not normal at $z_0\in D$. By Lemma 1, for $\alpha=\frac{\sum_{j=1}^{k}jn_j}{n+\sum_{j=1}^{k}n_j}$ there exist

1) a real number r, $0<r<1$,

2) points z_v, $|z_v|<r$, $z_v\to z_0$,

3) positive numbers $\rho_v,\rho_v\to0^+$,

4) functions f_v, $f_v \in \mathcal{F}$

such that

$$g_v(\xi) = \frac{f_v(z_v + \rho_v \xi)}{\rho_v^\alpha} \to g(\xi) \quad (3.1)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function all of whose zeros have multiplicity at least k and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$.

On the other hand,

$$(g_v^{(j)}(\xi))^{n_j} = ((\frac{f_v(z_v + \rho_v \xi)}{\rho_v^\alpha})^{(j)})^{n_j}$$
$$= \frac{1}{\rho_v^{n_j(\alpha - j)}} (f_v^{(j)})^{n_j}(z_v + \rho_v \xi)$$

for all $j = 1, \ldots, k$. Therefore, by the definition of α and by (3.1), we have

$$f_v^n(z_v + \rho_v\xi)(f_v')^{n_1}(z_v + \rho_v\xi)$$
$$\cdots (f_v^{(k)})^{n_k}(z_v + \rho_v\xi)$$
$$= g_v^n(\xi)(g_v'(\xi))^{n_1} \ldots (g_v^{(k)}(\xi))^{n_k}$$
$$\to g^n(\xi)(g'(\xi))^{n_1} \ldots (g^{(k)}(\xi))^{n_k} \quad (3.2)$$

spherically uniformly on compact subsets of $\mathbb{C} \setminus P$, where P denotes by the set of poles of g. We have $g^n(\xi)(g'(\xi))^{n_1} \ldots (g^{(k)}(\xi))^{n_k} \not\equiv a$. Indeed, if

$$g^n(\xi)(g'(\xi))^{n_1} \ldots (g^{(k)}(\xi))^{n_k} \equiv a. \quad (3.3)$$

Then g is a nonconstant entire function with order at most one. By Lemma 2, we get $g(\xi) = ce^{d\xi}$, where $d \neq 0$. This contradicts with (3.3). From (3.2) and Hurwtiz's theorem, we see that $g^n(\xi)(g'(\xi))^{n_1} \ldots (g^{(k)}(\xi))^{n_k} - a$ has at most one zero. This is a contradiction with Lemma 4 and Lemma 3. Hence, $\mathcal{F}$ is a normal family. □

References

[1] J. Clunie and W. K. Hayman (1966), "The spherical derivative of integral and meromorphic functions", *Comm. Math. Helv*, 40, pp. 117-148.

[2] X. C. Pang and L. Zalcman (1999), "On theorems of Hayman and Clunie", *New Zealand J. Math*, 28, pp. 71-75.

[3] C. C. Yang and P. C. Hu (1996), "On the value distribution of $ff^{(k)}$", *Kodai J. Math*, 19, pp. 157-167.

[4] L. Zalcman (1998), "Normal families: New perspective", *Bull. Amer. Mat. Soc*, 35, pp. 215-230.

Summary

MỘT SỐ TIÊU CHUẨN CHUẨN TẮC MỚI CHO HỌ CÁC HÀM PHÂN HÌNH LIÊN QUAN ĐẾN KẾT QUẢ CỦA PANG-ZALCMAN

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Bài báo quan tâm đến những lĩnh vực thú vị trong giải tích phức: Lý thuyết Nevanlinna và Họ chuẩn tắc Chúng tôi chứng minh một số tiêu chuẩn chuẩn tắc mới cho họ các hàm phân hình liên quan đến tiêu chuẩn chuẩn tắc của Pang-Zalcman [2]. Kết quả chính của chúng tôi được phát biểu như sau: Cho a là số phức khác không và $n \geq 2$ là số nguyên dương, $n_1, \ldots, n_{k-1}$ là các số nguyên không âm và n_k là số nguyên dương ($k \geq 1$). Cho $\mathcal{F}$ là họ các hàm phân hình trên miền D mà mọi

không điểm có bội ít nhất k mà $E_f = \{z : f^n(z)(f')^{n_1}(z)\cdots(f^{(k)})^{n_k}(z) - a = 0\}$ chứa nhiều nhất một điểm trong D với mọi $f \in \mathcal{F}$. Khi đó họ $\mathcal{F}$ chuẩn tắc trên D. Trong hiểu biết tốt nhất của chúng tôi, đây là kết quả mới bổ sung cho kết quả của Pang-Zalcman.
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