ZEROS OF THE DERIVATIVE OF A $p-$ADIC MEROMORPHIC FUNCTION

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ABSTRACT

The theorem of Picard in its simplest form asserts that every nonconstant function $f(z)$, meromorphic in the plane, assumes there all complex values w with the possible exception of two. A value w which is not assumed by $f(z)$ will be called a Picard exceptional value. In 1959, Hayman [4] created an important research subject in considering the value distributions of differential polynomials, that is if f is a transcendental meromorphic function and $n \in \mathbb{N}$, then $f'f^n$ takes every finite nonzero value infinitely often. The Hayman conjecture implies that the finite Picard exceptional value of $f'f^n$ may only be zero. Using techniques of Nevanlinna theory, we showed that for a transcendental meromorphic function f in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value $\mathbb{K}$ and let $k \in \mathbb{N}^*$, then the function $(f^n)^{(k)}$ takes every value $b \in \mathbb{K}, b \neq 0$ infinitely many times if $n \geq 4$, which generalizes the related result due to Ojeda [8] for some differential polynomials of k-th derivative.

Keywords: *Differential polynomial, value distribution, non-Archimedean, $p-$adic meromorphic function, exceptional values.*

1 Introduction and main result

Now let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. and f be a nonconstant meromorphic function on $\mathbb{K}$. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{C}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$. Let $f \in \mathcal{M}(\mathbb{K})$ such that $f(0) \neq 0, \infty$. We denote by $S(r, f)$ any function satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$ outside of a possible exceptional set with finite measure. we call *quasi-exceptional value* for a transcendental meromorphic function f in $\mathbb{K}$ a value $b \in \mathbb{K}$ such that $f - b$ has finitely many zeros.

In 1926, as an application of the celebrated Nevanlinna's value distribution theory of meromorphic functions, Nevanlinna proved that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ cannot have the same inverse images ignoring multiplicities for five distinct values, and f is a Möbius transformation of g if they have the same inverse images counting multiplicities for four distinct values. In general, the number four can not be re-

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duced.

In 1959, Hayman [4] created an important research subject in considering the value distributions of differential polynomials, that is if f is a transcendental meromorphic function and $n \in \mathbb{N}$, then $f'f^n$ takes every finite nonzero value infinitely often. He conjectured it should be hold for any n. This conjecture has been solved by Hayman [4] for $n \geq 3$, by Mues [7] for $n = 2$, by Bergweiler and Eremenko [1] and Chen and Fang [2] for $n = 1$. The Hayman conjecture implies that the finite Picard exceptional value of $f'f^n$ may only be zero.

In recent years, similar problems for functions in non-Archimedean fields are investigated (see, e.g., [3, 6, 8]). In [8] J. Ojeda proved that, for a transcendental meromorphic function f in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value $\mathbb{K}$; the function $f'f^n - 1$ has infinitely many zeros if $n \geq 2$. Note that $f'f^n = \frac{1}{n+1}(f^{n+1})'$. A natural generalization is considering differential polynomials of k-th derivative instead of the first derivative. In this paper, we will reach to the direction by considering differential polynomials of k-th derivative $[f^n]^{(k)}$. Our results as follows.

Main Theorem. *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let $k \in \mathbb{N}^*$. Then, for every $n \in \mathbb{N}$ and $n \geq 4$, $(f^n)^{(k)}$ takes every value $b \in \mathbb{K}, b \neq 0$ infinitely many times.*

As an immediate consequence of Main Theorem, we obtain a special case as following.

Corollary 2.([8]) *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental. Then, for every $b \in \mathbb{K}$ different from 0, $f^3f' - b$ has infinitely many zeros.*

2 Preliminary on Nevanlinna's Theory

We recall some standard definitions and results in Nevanlinna theory. Let f be a meromorphic function on $\mathbb{K}$. Let $n(t, f)$ be the number of poles of $f(z)$ in $|z| \leq t$ each counted with correct multiplicity and $\overline{n}(t, f)$ the number of poles of $f(z)$ in $|z| \leq t$, where each multiple pole is counted only once. The *counting function* of poles is defined as follows

$$N(r, f) := \int_0^r [n(t, f) - n(0, f)]\frac{dt}{t} + n(0, f)\log r,$$

with similar definition for $\overline{N}(r, f)$. The *proximity function* and *characteristic function* are defined respectively as follows

$$m(r, f) := \log^+ \mu(r, f) = \max\{0, \mu(r, f)\}$$

$$T(r, f) := m(r, f) + N(r, f).$$

The logarithmic derivative lemma can be stated as follows.

Lemma 2.1. *Let f be a non-constant meromorphic function on $\mathbb{K}$. Then*

$$m(r, \frac{f'}{f}) = O(1)$$

as $r \to \infty$ outside a subset of finite measure.

We state the first and second fundamental theorem in $p-$adic Nevanlinna theory (see e.g. [8]):

Theorem 2.2. *Let $f(z)$ be a meromorphic function and $c \in \mathbb{K}$. Then*

$$T(r, \frac{1}{f - c}) = T(r, f) + O(1).$$

Theorem 2.3. *Let $a_1, \cdots, a_q$ be a set of distinct numbers in $\mathbb{K}$. Let f be a*

non-constant meromorphic function on $\mathbb{K}$. *Then, the inequality*

$$(q-1)T(r,f) \leq \overline{N}(r,f) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) - \log r + S(r,f),$$

as $r \to \infty$ *outside a subset of finite measure.*

3 Proof of Main Theorem

For the proof of our result, we first discuss the following lemmas

Lemma 3.1 (Milloux's inequality)([5]) *Let* $f \in \mathcal{M}(\mathbb{K})$ *be non-constant. Then*

$$T(r,f) \leq \overline{N}(r,f) + N(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)}-1}) - N(r, \frac{1}{f^{(k+1)}}) - \log r + S(r,f).$$

Lemma 3.2. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental. Then, for* $k \geq 2$ *be integer and* $\epsilon > 0$,

$$N(r, \frac{1}{f}) \leq 2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) + \epsilon T(r,f) + S(r,f). \quad (1)$$

Proof. We will follow Wang's proof in [9]. If f has only finitely many zeros and poles, then (1) holds.

Now, we assume that f has infinitely many zeros and poles. For given $k \geq 2$ and $\epsilon > 0$, we choose an $m \in \mathbb{Z}$ such that

$$k^2 + 2 - k < m\epsilon.$$

Set

$$W = W(f, zf, \ldots, z^m f, f', zf', \ldots, z^m f'),$$

where $W(f_1, f_2, \ldots, f_n)$ denotes the Wronskian of $f_1, f_2, \ldots, f_n$. Then, we have $W \not\equiv 0$. Otherwise, if $W \equiv 0$, then there exists numbers (not all of them zero) $\alpha_i, \beta_i (i = 0, \ldots, m)$ in $\mathbb{K}$ such that

$$\left(\sum_{i=0}^{m} \alpha_i z^i\right) f + \left(\sum_{i=0}^{m} \beta_i z^i\right) f' \equiv 0. \quad (2)$$

Since $\left(\sum_{i=0}^{m} \alpha_i z^i\right)^2 + \left(\sum_{i=0}^{m} \beta_i z^i\right)^2 \not\equiv 0$, hence by (2) we see that f has only finitely many zeros and poles. This is a contradiction to the assumption. Evidently, from the $(k+m+1)-$row to the last row, each term in these rows is a differential polynomial in f with coefficients $cz^i (0 \leq i \leq m, c \in \mathbb{K})$ without derivatives of f of order less than k. Set

$$\Phi = \frac{W}{f^{m+k}(f^{(k)})^{m+2-k}}.$$

By the logarithmic derivative lemma, we show that $m(r, \Phi) = S(r,f)$.

Applying Jensen's formula to Φ, we have

$$N(r, \frac{1}{\Phi}) = \log \mu(r, \Phi) + N(r, \Phi) + O(1) \leq N(r, \Phi) + S(r,f),$$

and hence

$$\begin{aligned} 0 &\leq N(r,\Phi) - N(r, \frac{1}{\Phi}) + S(r,f) \\ &= N\left(r, \frac{W}{f^{m+k}(f^{(k)})^{m+2-k}}\right) \\ &\quad - N\left(r, \frac{f^{m+k}(f^{(k)})^{m+2-k}}{W}\right) \\ &\quad + S(r,f) \\ &= \log \mu\left(\rho_0, \frac{W}{f^{m+k}(f^{(k)})^{m+2-k}}\right) \\ &\quad - \log \mu\left(r, \frac{W}{f^{m+k}(f^{(k)})^{m+2-k}}\right) \\ &\quad + S(r,f). \end{aligned}$$

Thus

$$\begin{aligned} 0 \le & N(r,W) - N(r,\frac{1}{W}) \\ & - N(r, f^{m+k}(f^{(k)})^{m+2-k}) \\ & + N\Big(r, \frac{1}{f^{m+k}(f^{(k)})^{m+2-k}}\Big) \\ & + S(r,f). \end{aligned} \tag{3}$$

If z_0 is a zero of f of order p, then by substituting f by their Taylor series representations at point z_0 in W and by a property of Wronskian, we have z_0 is a zero point of W of order at least $(2m+2)(p-1)$. Therefore, we get

$$N(r,\frac{1}{W}) \ge (2m+2)(N(r,\frac{1}{f}) - \overline{N}(r,\frac{1}{f})). \tag{4}$$

By a similar computation, we obtain

$$N(r,W) \le (2m+2)(N(r,f) - \overline{N}(r,f)). \tag{5}$$

From (3), (4), (5) and $k \ge 2$, we obtain

$$\begin{aligned} mN(r,\frac{1}{f}) \le & 2m\overline{N}(r,\frac{1}{f}) + mN(r,\frac{1}{f^{(k)}}) \\ & + (k^2-k+2)T(r,f) + N(r,\frac{1}{f^{(k)}})) \\ & + (2-k)(m\overline{N}(r,f) + S(r,f) \\ \le & 2m\overline{N}(r,\frac{1}{f}) + mN(r,\frac{1}{f^{(k)}}) \\ & + (k^2-k+2)T(r,f) + S(r,f), \end{aligned}$$

this completes of proof.

Proof of main theorem. Suppose $f \in \mathcal{M}(\mathbb{K})$ to be transcendental and suppose $b \ne 0$ to be a quasi-exceptional value of $f^{(k)}$. Without the loss of generality, we can suppose $b = 1$. Applying Lemma 3.1 and lemma 3.2 with $\epsilon = \frac{1}{2n}$, we obtain

$$\begin{aligned} T(r,f^n) \le & \overline{N}(r,f^n) + N(r,\frac{1}{f^n}) \\ & + \overline{N}(r,\frac{1}{(f^n)^{(k)}-1}) - N(r,\frac{1}{(f^n)^{(k+1)}}) \\ & - \log r + S(r,f) \\ \le & 2\overline{N}(r,\frac{1}{f^n}) + \overline{N}(r,\frac{1}{(f^n)^{(k)}-1}) \\ & + \overline{N}(r,f) + \epsilon T(r,f^n) - \log r + S(r,f). \end{aligned}$$

Hence, we get

$$\begin{aligned} \Big(n-\frac{1}{2}\Big)T(r,f) \le & \overline{N}(r,f) + 2\overline{N}(r,\frac{1}{f}) \\ & + \overline{N}(r,\frac{1}{(f^n)^{(k)}-1}) - \log r + S(r,f), \end{aligned}$$

which gives

$$\begin{aligned} \Big(n-\frac{7}{2}\Big)T(r,f) \le & \overline{N}(r,\frac{1}{f^{(k)}-1}) - \log r \\ & + S(r,f). \end{aligned} \tag{6}$$

Since the number of zeros of $f^{(k)} - 1$ is q, taking multiplicity into account, we have

$$\overline{N}(r,\frac{1}{f^{(k)}-1}) \le q\log r + O(1),$$

hence by (6) we obtain

$$\Big(n-\frac{7}{2}\Big)T(r,f) \le O(\log r),$$

which contradicts the hypothesis $n \ge 4$. The proof is complete.

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Tóm tắt

CÁC KHÔNG ĐIỂM CỦA ĐẠO HÀM CỦA MỘT HÀM PHÂN HÌNH P–ADIC

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Định lí Picard ở dạng đơn giản nhất khẳng định rằng mỗi hàm phân hình khác hằng f trên mặt phẳng phức, nhận tất cả các giá trị phức w có thể trừ ra hai giá trị. Nếu f không nhận giá trị w thì w được gọi là giá trị bỏ được Picard. Năm 1959, Hayman [4] đã tạo ra một đối tượng nghiên cứu quan trọng bằng việc xem xét các giá trị phân bố của các đa thức vi phân, tức là nếu f là một hàm phân hình siêu việt và $n \in \mathbb{N}$, thì $f'f^n$ nhận mỗi giá trị hữu hạn khác không vô hạn lần. Giả thuyết Hayman cho thấy rằng giá trị bỏ được Picard hữu hạn của $f'f^n$ chỉ có thể là 0. Sử dụng các kỹ thuật trong lý thuyết Nevanlinna, chúng tôi chứng minh rằng nếu với một hàm phân hình siêu việt f trong một trường đóng đại số, đầy đủ với một giá trị tuyệt đối không Acsimet $\mathbb{K}$ và cho $k \in \mathbb{N}^*$, thì hàm $(f^n)^{(k)}$ nhận mỗi giá trị $b \in \mathbb{K}, b \neq 0$ vô hạn lần nếu $n \geq 4$. Kết quả này của chúng tôi là một tổng quát kết quả của Ojeda [8] cho đa thức vi phân trong trường hợp đạo hàm cấp cao.

Từ khóa: Đa thức đạo hàm, phân bố giá trị, không Acsimet, hàm phân hình p–adic, các giá trị ngoại trừ.

Ngày nhận bài:06/3/2017; Ngày phản biện:14/3/2017; Ngày duyệt đăng: 31/5/2017.