IMPLICIT ITERATION METHODS HILBERT SPACES

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Abstract. In this paper, we introduce new implicit and an explicit iteration methods based on Krasnoselskii-Mann iteration method and a contraction for finding a fixed point of a nonexpansive self-mapping of a closed convex subset of a real Hilbert space.

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1. INTRODUCTION

Let C be a nonempty closed and convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let T be a nonexpansive self-mapping of C, i.e., $T : CC$ and $\|Tx - Ty\| \leq \|x - y\|$. Denote the set of fixed points of T by $F(T)$, i.e., $F(T) := \{x \in C : x = Tx\}$, and the projection of $x \in H$ onto C by $P_C(x)$. In this paper, we assume that $F(T) \neq \emptyset$.

Theorem 1.1. [1] *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive self-mapping of C such that $F(T) \neq \emptyset$. Let f be a contraction of C with a constant $\tilde{\alpha} \in [0, 1)$ and let $\{x_k\}$ be a sequence generated by: $x_1 \in C$ and*

$$x_k = \frac{\lambda_k}{1+\lambda_k} f(x_k) + \frac{1}{1+\lambda_k} Tx_k, \quad k \geq 1, \tag{1.1}$$

$$x_{k+1} = \frac{\lambda_k}{1+\lambda_k} f(x_k) + \frac{1}{1+\lambda_k} Tx_k, \quad k \geq 1, \tag{1.2}$$

where $\{\lambda_k\} \subset (0, 1)$ satisfies the following conditions:

(L1) $\lim_{k\to\infty} \lambda_k = 0$;

(L2) $\sum_{k=1}^{\infty} \lambda_k = \infty$; *and*

(L3) $\lim_{k\to\infty}\left|\frac{1}{\lambda_{k+1}} - \frac{1}{\lambda_k}\right| = 0$.

Then, $\{x_k\}$ defined by (1.2) converges strongly to $p^* \in F(T)$, where $p^* = P_{F(T)}f(p^*)$ and $\{x_k\}$ defined by (1.1) converges to p^* only under condition (L1).

Note that $p^* = P_{F(T)}f(p^*)$ is equivalent to the following variational inequality:

$$p^* \in F(T) : \langle p^* - f(p^*), p - p^*\rangle \geq 0 \quad \forall p \in F(T). \tag{1.3}$$

It is easy to see that the mapping $I - f$, where I denotes the identity mapping in H, is $(1 + \tilde{\alpha})$-Lipschitsz continuous and $(1 - \tilde{\alpha})$-strongly monotone.

In this paper, we propose some new modifications of (1.1) and (1.2) that are the implicit algorithm:

$$x_t = T^t x_t, \quad T^t := T_1^t T_0^t \quad \text{and} \quad T^t := T_0^t T_1^t, \quad t \in (0,1), \tag{1.4}$$

where T_i^t are defined by

$$\begin{aligned} T_0^t &= (1-\lambda_t\mu)I + \lambda_t\mu f, \\ T_1^t &= (1-\beta_t)I + \beta_t T, \end{aligned} \tag{1.5}$$

where f is a contraction with a constant $\tilde{\alpha} \in [0,1)$, $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$ and the parameters $\{\lambda_t\} \subset (0,1)$ and $\{\beta_t\} \subset (\alpha,\beta)$ for all $t \in (0,1)$ and some $\alpha, \beta \in (0,1)$ satisfying the following condition: $\lambda_t \to 0$ as $t \to 0$.

We formulate the following facts for the proof of our results.

Lemma 1.1 [2]. (i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$ *and for any fixed* $t \in [0,1]$

(ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - (1-t)t\|x-y\|^2, \quad \forall x, y \in H.$

Lemma 1.2 [3]. $\|T^\lambda x - T^\lambda y\| \le (1-\lambda\tau)\|x-y\|$ *for a fixed number* $\mu \in (0, 2\eta/L^2), \lambda \in (0,1)$, *where* $\tau = 1 - \sqrt{1-\mu(2\eta - \mu L^2)} \in (0,1)$,

$$T^\lambda x = (I - \lambda\mu F)x$$

and F *is* L*-Lipschitz continuous and* η*-strongly monotone.*

Lemma 1.3 (*Demiclosedness Principle* [4]). *Assume that* T *is a nonexpansive self-mapping of a closed convex subset* K *of a Hibert space* H. *If* T *has a fixed point, then* $I - T$ *is demiclosed; that is, whenever* $\{x_k\}$ *is a sequence in* K *weakly converging to some* $x \in K$ *and the sequence* $\{(I-T)x_k\}$ *strongly converges to some* y, *it follows that* $(I-T)x = y$.

2. IMPLICIT ITERATION METHODS

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $f : C \to C$ *be a contraction with a coefficient* $\tilde{\alpha} \in [0,1)$. *Let* T *be a nonexpansive self-mapping of* C *such that* $F(T) \ne \emptyset$. *Let* $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$. *Then, the net* $\{x_t\}$ *defined by (1.4)-(1.5) converges strongly to the unique element* p^* *in (1.3).*

Proof. First, we consider the case that $T^t = T_1^t T_0^t$. Clearly, T^t is a self-mapping of C and $T_0^t = I - \lambda_t\mu F$ where $F = I - f$. By (1.5) and Lemma 1.2, we have that

$$\begin{aligned} \|T^t x - T^t y\| &= \|T_1^t T_0^t x - T_1^t T_0^t y\| \\ &= \|(1-\beta_t)T_0^t x + \beta_t T_0^t x - [(1-\beta_t)T_0^t y + \beta_t T_0^t y]\| \\ &= \|(1-\beta_t)(T_0^t x - T_0^t y) + \beta_t(T_0^t x - \beta_t T_0^t y)\| \\ &\le \|T_0^t x - T_0^t y\| \\ &\le (1-\lambda_t\tau)\|x-y\| \quad \forall x, y \in C, \end{aligned}$$

where τ is defined in Lemma 1.2 with $\eta = 1 - \tilde{\alpha}$ and $L = 1 + \tilde{\alpha}$. So, T^t is a contraction of C. By Banach's Contraction Principle, there exists a unique element $x_t \in C$ such that $x_t = T^t x_t$ for all $t \in (0,1)$.

Next, we show that $\{x_t\}$ is bounded. Indeed, it is easy to see that T_1^t also is nonexpansive with $T_1^t p = p$ for a fixed point $p \in F(T)$, and hence

$$\begin{aligned}\|x_t - p\| &= \|T^t x_t - p\| = \|T_1^t T_0^t x_t - T_1^t p\| \\ &\leq \|T_0^t x_t - p\| \\ &= \|T_0^t x_t - T_0^t p - \lambda_t \mu F(p)\| \\ &\leq (1 - \lambda_t \tau)\|x_t - p\| + \lambda_t \mu \|F(p)\|.\end{aligned}$$

Therefore,

$$\|x_t - p\| \leq \frac{\mu}{\tau}\|F(p)\|$$

that implies the boundedness of $\{x_t\}$ Put $y_t = T_0^t x_t$. So, the nets $\{F(x_t)\}, \{y_t\}$ also are bounded. Moreover, we also have from (1.4)-(1.5) that

$$x_t = (1 - \beta_t)y_t + \beta_t T y_t, \tag{2.1}$$

and

$$\begin{aligned}\|x_t - p\|^2 &= \|(1 - \beta_t)y_t + \beta_t T y_t - p\|^2 \\ &\leq \|y_t - p\|^2 = \|(I - \lambda_t \mu F)x_t - p\|^2 \\ &= \|x_t - p\|^2 - 2\lambda_t \mu \langle F(x_t), x_t - p\rangle + \lambda_t^2 \mu^2 \|F(x_t)\|^2\end{aligned}$$

Thus,

$$(1 - \tilde{\alpha})\|x_t - p\|^2 + \langle F(p), x_t - p\rangle \leq \frac{\lambda_t \mu}{2}\|F(x_t)\|^2. \tag{2.2}$$

Further, we prove that $\|x_t - Tx_t\| \to 0$, as $t \to 0$. Since $\|y_t - x_t\| = \lambda_t \mu \|F(x_t)\| \to 0$ as $t \to 0$, because $\lambda_t \to 0$ and $\{F(x_t)\}$ is bounded. So, we shall prove that $\|y_t - Ty_t\| \to 0$, as $t \to 0$.

Let $\{t_k\} \subset (0, 1)$ be an arbitrary sequence converging to zero as $k \to \infty$ and $x_k := x_{t_k}, y_k := y_{t_k} = (1 - \lambda_k \mu)x_k + \lambda_k \mu f(x_k)$, where $\lambda_k = \lambda_{t_k}$. We have to prove that $\|y_k - Ty_k\| \to 0$. Let $\{x_l\}$ be a subsequence of $\{x_k\}$ and let $\{x_{k_j}\}$ be a subsequence of $\{x_l\}$ such that

$$\begin{aligned}&\limsup_{k\to\infty} \|y_k - Ty_k\| = \lim_{l\to\infty} \|y_l - Ty_l\|, \\ &\limsup_{l\to\infty} \|x_l - p\| = \lim_{j\to\infty} \|x_{k_j} - p\|.\end{aligned}$$

From (2.1) and Lemma 1.1, it implies that

$$\begin{aligned}\|x_{k_j} - p\|^2 &= \|(1 - \beta_{k_j})(y_{k_j} - p) + \beta_{k_j}(Ty_{k_j} - p)\|^2 \\ &\leq \|y_{k_j} - p\|^2 = \|(I - \lambda_{k_j}\mu F)x_{k_j} - p\|^2 \\ &\leq \|x_{k_j} - p\|^2 + 2\lambda_{k_j}\mu \|F(x_{k_j})\| \|y_{k_j} - p\|.\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} \|x_{k_j} - p\| = \lim_{j\to\infty} \|y_{k_j} - p\|, \tag{2.3}$$

since $\lambda_{k_j} \to 0$ and $\{F(x_{k_j})\}, \{y_{k_j}\}$ are bounded. Again, by Lemma 1.1, we have that

$$\begin{aligned}\|x_{k_j} - p\|^2 &= (1 - \beta_{k_j})\|y_{k_j} - p\|^2 + \beta_{k_j}\|Ty_{k_j} - p\|^2 \\ &\quad - \beta_{k_j}(1 - \beta_{k_j})\|y_{k_j} - Ty_{k_j}\|^2 \\ &\leq (1 - \beta_{k_j})\|y_{k_j} - p\|^2 + \beta_{k_j}\|y_{k_j} - p\|^2 \\ &\quad - \beta_{k_j}(1 - \beta_{k_j})\|y_{k_j} - Ty_{k_j}\|^2\end{aligned}$$

Then, we have

$$\alpha(1-\beta)\|y_{k_j}-Ty_{k_j}\|^2 \le \|y_{k_j}-p\|^2-\|x_{k_j}-p\|^2.$$

This together with (2.3) implies that

$$\lim_{j\to\infty}\|y_{k_j}-Ty_{k_j}\|^2=0.$$

Consequently, $\|y_t-Ty_t\|\to 0$ as $t\to 0$.

Let $\{x_k\}$ be any sequence of $\{x_t\}$ converging weakly to $\tilde{p}$ as $k\to\infty$. Then, $\|x_k - Tx_k\|\to 0$. By Lemma 1.3, we have $\tilde{p}\in F(T)$ and from (2.2), it follows that

$$\langle F(p),p-\tilde{p}\rangle \ge 0 \quad \forall p\in F(T).$$

Since $p,\tilde{p}\in F(T)$ which is a closed convex subset, by replacing p by $tp+(1-t)\tilde{p}$ in the last inequality, dividing by t and taking $t\to 0$ in the just obtained inequality, we obtain

$$\langle F(\tilde{p}),p-\tilde{p}\rangle \ge 0 \quad \forall p\in Fix(T).$$

The uniqueness of p^* in (1.3) guarantees that $\tilde{p}=p^*$. Again, replacing p in (2.2) by p^*, we obtain the strong convergence for $\{x_t\}$.

The case that $T^t=T_0^tT_1^t$ is proved similarly. T^t is also a contraction So, there exists a unique x_t for each $t\in(0,1)$ such that $x_t=T_0^tT_1^tx_t$ and $\{x_t\}$ is bounded. Put $y_t=T_1^tx_t$. Then, the nets $\{F(y_t)\}$ and $\{y_t\}$ are also bounded. Further, (2.1) and (2.2) are replaced by

$$x_t=(I-\lambda_t\mu F)y_t, \quad \text{and}$$

$$(1-\tilde{\alpha})\|y_t-p\|^2+\langle F(p),y_t-p\rangle \le \frac{\lambda_t\mu}{2}\|F(y_t)\|^2,$$

respectively. From the last equality, the boundedness of $\{F(y_t)\}$ and $\lambda_t\to 0$, it follows that $\|x_t-y_t\|\to 0$ as $t\to 0$. Next, we prove that $\|x_t-Tx_t\|\to 0$, as $t\to 0$. Let $\{t_k\}\subset(0,1)$ be an arbitrary sequence converging to zero as $k\to\infty$ and $x_k:=x_{t_k}$. We have to prove that $\|x_k-Tx_k\|\to 0$. Let $\{x_l\}$ be a subsequence of $\{x_k\}$ and let $\{x_{k_j}\}$ be a subsequence of $\{x_l\}$ such that

$$\limsup_{k\to\infty}\|x_k-Tx_k\|=\lim_{l\to\infty}\|x_l-Tx_l\|,$$
$$\limsup_{l\to\infty}\|x_l-p\|=\lim_{j\to\infty}\|x_{k_j}-p\|.$$

Further, the process of proof is similar as in the case that $T^t=T_1^tT_0^t$. This completes the proof.

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Phương pháp lặp ẩn trong không gian Hilbert

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Tóm tắt

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