SHRINKING PROJECTION AND MODIFIED HYBRID DOUGLAS-RACHFORD SPLITTING METHOD FOR MONOTONE INCLUSIONS IN HILBERT SPACES

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ABSTRACT: In this paper, in order to find a zero for a monotone inclusion in Hilbert spaces, we introduce shrinking projection and hybrid Douglas-Rachford splitting methods, based on the Takahashi et. al shrinking method for nonexpansive mapping, hybrid method in mathematical programming and Douglas-Rachford splitting method.

Keywords: *Nonexpansive mapping; fixed point; shrinking projection; hybrid splitting method; monotone mapping.*

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1 Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let T be a maximal monotone nonlinear mapping in H, that is, T is monotone, i.e., T satisfies the condition: $\langle u - v, x - y \rangle \geq 0$ for $u \in T(x)$, $v \in T(y)$ with x, y in the domain of T, and, in addition, the graph of T is not included in the graph of any other monotone mapping.

The problem, studied in this paper, is to find a zero for the monotone inclusion

$$0 \in T(x), \quad T = A + B, \tag{1.1}$$

where A and B are maximal monotone in H.

A fundamental algorithm for finding a root of a maximal monotone mapping T is the proximal point algorithm: $x_1 \in H$ and

$$x_{k+1} = J^T_{r_k} x_k, \quad k = 0, 1, \cdots, \tag{1.2}$$

where $J^T_{r_k} = (I + r_k T)^{-1}$ and $\{r_k\} \subset (0, \infty)$ and I denotes the identity mapping in H. This algorithm was firstly introduced by Martinet [1]. In [2], Rockafellar proved that if $\liminf_{k\to\infty} r_k > 0$ and $T^{-1}0 \neq \emptyset$, then the sequence $\{x_k\}$, defined by (1.2), converges weakly to a solution of (1.1). In [3], Guler showed that it converges only weakly in infinite dimentional space H. Note that, in many cases, for a fixed $\gamma > 0$, $I + \gamma T$ is hard to invert, but $I + \gamma A$ and $I + \gamma B$ are easier to invert than $I + \gamma T$, where $T = A + B$ and A, B are two maximal monotone mappings. Splitting methods for problem (1.1) are algorithms that do not attempt to evaluate the resolvant $(I + \gamma T)^{-1}$, but instead perform a sequence of calculations involving only the resolvants $(I + \gamma A)^{-1}$ and $(I + \gamma B)^{-1}$. Such an approach is inspired by well-established techniques from numerical linear algebra (see, [4]). Monotone

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operator splitting algorithms have extensive literature, all of which can essentially be devided into four principal classes: forward-backward class (see, [5]-[8]) double-backward (see, [7] and [9]), Peaceman-Rachford (see, [10]), and Douglas-Rachford (see [10]-[13]). To the best of our knowledge, all of Douglas-Rachford splitting methods only produce weakly convergent iterative sequences. Moreover, the weak cluster points of these weakly convergent iterative sequences only solve some fixed points equation, but not the original monotone inclusion problem. For example, see [8].

Theorem 1.1. *If A, B are maximal monotone mappings in a Hilbert space H and the solution set of $0 \in Ax + Bx$ is nonempty. Let λ_k be a sequence in $[0,2]$ such that $\sum_{k\in\mathbb{N}} \lambda_k(2-\lambda_k) = \infty$, let $\gamma \in \mathbb{R}_{++}$, and let $x_0 \in H$. Set*

$$
\begin{aligned}
y_k &= J_\gamma^B x_k,\\
z_k &= J_\gamma^A(2y_k - x_k),\\
x_{k+1} &= x_k + \lambda_k(z_k - y_k).
\end{aligned}
\tag{1.3}
$$

Then, there exists $x \in \mathrm{Fix}(R_\gamma^A R_\gamma^B)$ $x \in \mathrm{Fix}(R_\gamma^A R_\gamma^B)$ such that:
(i) $J_\gamma^B x \in Zer(A+B)$;
(ii) (x_k) converges weakly to x;
(iii) (y_k) converges weakly to $J_\gamma^B x$;
(iv) (z_k) converges weakly to $J_\gamma^B x$,

where $R_\gamma^D = 2J_\gamma^D - I$ for mapping D in H.

Recently, Zhang and Cheng proved [14] that $J_\gamma^B x_k$ converges weakly to a zero of $A+B$, assumed to be maximal monotone With the additional assumption, they also obtained strong convergent result, by combining the method with Haugazeau's method [15].

Clearly, method (1.3) can be rewritten in the form

$$x_{k+1} = S(1/2)x_k$$

where

$$S(\alpha) = \alpha T + (1-\alpha)I,\ T = (R_\gamma^A R_\gamma^B),$$

with nonexpansive mappings T, since R_γ^A and R_γ^B are nonexpansive (see [9]). We also know in [8] that $J_\gamma^B \mathrm{Fix}(T) = \mathrm{Zer}(A+B)$. It is well-known know that a sequence $\{x_k\}$, generated by Krasnoselski-Mann method:

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T x_k,$$

only converges weakly to an element of $\mathrm{Fix}(T)$, assumed to be nonempty, if $\alpha_k \in [0,1]$ such that

$$\sum_{k\in\mathbb{N}} \alpha_k(1-\alpha_k) = +\infty.$$

In order to obtain convergent result, using Solodov and Svaiter's approach [16], Nakajo and Takahashi, in [17], introduced the following algorithm:

$$
\begin{aligned}
&x_1 \in C,\ \text{any element}\\
&y_k = \alpha_k x_k + (1-\alpha_k)Tx_k,\\
&C_k = \{z \in C : \|z - y_k\| \le \|z - x_k\|\},\\
&Q_k = \{z \in C : \langle x_k - z, x_1 - x_k\rangle \ge 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_1),\ \ k = 1, 2, \cdots,
\end{aligned}
\tag{1.4}
$$

where $\{x_k\} \subset [0,1]$ satisfies $\limsup_{k\to\infty} \alpha_k < 1$. Developing the idea, in [18], in order to find a fixed point of a nonexpansive mapping T on a closed convex subset C in H, Takahashi introduced an alternative projection method:

$$\begin{aligned} x_0 &= x \in C := C_0, \\ y_k &= \alpha_k x_k + (1-\alpha_k)Tx_k, \\ C_{k+1} &= \{z \in C_k : \|y_k - z\| \le \|x_k - z\|\}, \\ x_{k+1} &= P_{C_{k+1}}(x), \ \ k = 1, 2, \cdots, \end{aligned} \tag{1.5}$$

which is called the shrinking projection method. They proved the strong convergence of this sequence to $P_{Fix(T)}x$, if $\alpha_k \in [0,a]$ for all $k \ge 0$ and $a \in (0,1)$. See, also [19].

In this paper, motivated by the results (1.3) - (1.5), without assuming $A+B$ to be maximal monotone, we introduce two new iteration methods. The first method is constructed as follows:

$$\begin{aligned} x_0 &\in H_0 = H, \ \textit{any element}. \\ y_k &= J_\gamma^B x_k, \\ z_k &= J_\gamma^A(2y_k - x_k), \\ v_k &= x_k + \epsilon_k(z_k - y_k), \\ H_{k+1} &= \{z \in H_k : \|v_k - z\|^2 \le \|p_k - z\|^2\}, \\ x_{k+1} &= P_{H_{k+1}} f(p_k), \ \ k \ge 0, \end{aligned} \tag{1.6}$$

where f is a Meir-Keeler contraction (see, [20]) in H and sequence $\{\epsilon_k\}$ satisfies the conditions $0 < \underline{\epsilon} \le \epsilon_k < 2$. As in [21], the second method is discribed in the form:

$$\begin{aligned} x_0 &\in H, \ \textit{any element}, \\ y_k &= J_\gamma^B x_k, \\ z_k &= J_\gamma^A(2y_k - x_k), \\ v_k &= x_k + \epsilon_k(z_k - y_k), \\ H_k &= \{z \in H_k : \|v_k - z\| \le \|x_k - z\|\}. \\ W &= \begin{cases} H, & if\ k = 0 \\ \{z \in W_{k-1} : \langle f(x_{k-1}) - x_k, x_k - z\rangle \ge 0\}, & if\ k \ge 1, \end{cases} \\ x_{k+1} &= P_{H_k \cap W_k} f(x_k), \ k \ge 0. \end{aligned} \tag{1.7}$$

Recall that mapping f in a metric space X with distance $d(x,y)$ is said to be a Meir-Keeler contraction, if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that for each $x, y \in X$, $\epsilon \le d(x,y) \le \epsilon + \delta$ implies that $d(f(x), f(y)) \le \epsilon$.

We will make use the following well-known results.

Lemma 1.1. (see [8]). *Let C be a non-empty, closed and convex subset in H. Then, we have:*

$$z = P_C(x) \Leftrightarrow \langle z - x, u - z\rangle \ge 0 \quad \forall u \in C, \forall x \in H$$

where $P_C(x)$ denotes the metric projection of x onto C.

Proposition 1.1. (see, [8], Corollary 25.5). *Let m be an integer such that $m \ge 2$, set $I = \{1, 2, \cdots, m\}$, and let $(A_i)_{i\in I}$ be maximally monotone mappings from a Hilbert* *space H to 2^H. For every $i \in I$, let $(x_{i,k}, u_{i,k})_{k\geq 1}$ be a sequence in the Gph A_i and let $(x_i, u_i) \in H \times H$. Suppose that*

$$\sum_{i\in I} u_{i,k} \to 0, \text{ and } x_{i,k} \overset{w}{\to} x_i;\ u_{i,k} \overset{w}{\to} u_i;\ m x_{i,k} - \sum_{j\in I} x_{j,k} \to 0.$$

Then, there exits $x \in zer \sum_{i\in I} A_i$ such that the following hold:

(i) $x = x_1 = \cdots = x_m$;

(ii) $\sum_{i\in I} u_i = 0$;

(iii) $\forall i \in I, (x_i, u_i) \in Gph\, A_i$.

2 Main Results

We prove the following results.

Theorem 2.1. *If A, B are maximal monotone mappings in Hilbert H such that the solution set of $0 \in Ax + Bx$ is nonempty, then the sequence $\{x_k\}$, defined by (1.6) with $0 < \underline{\epsilon} \leq \epsilon_k < 2$ converges strongly to some u_* such that $J_\gamma^B x_k$ converges to $J_\gamma^B u_* \in Zer(A+B)$.*

Proof. i) *We prove that x_k is defined for all $k \geq 0$.*

Clearly, H_k is a closed and convex subset in H for all $k \geq 0$. We prove that $H_k \neq \emptyset$. As mentioned above, J_γ^B Fix(T) = Zer$(A + B) \neq \emptyset$, and hence Fix$(T) \neq \emptyset$. On the other hand,

$$v_k = (1 - \epsilon/2)x_k + \epsilon_k T x_k/2. \tag{2.1}$$

Therefore, for any $p \in$ Fix(T), we have

$$\|v_k - p\| \leq (1 - \epsilon/2)\|x_k - p\| + \epsilon_k\|Tx_k - Tp\|/2 \leq \|x_k - p\|,$$

which implies that $H_k \neq \emptyset$. It means that x_k is well defined for each $k \geq 0$.

ii) *We show that there exists an element $\widetilde{p} \in H$ such that $x_k \to \widetilde{p}$ as $k \to \infty$.*

Consider the mapping $U_* = P_{\cap_{i\geq 0} H_i} f$. Since P_C is nonexpansive for any closed and convex subset C and f is a Meir-Keeler contraction, the mapping U_* is also a Meir-Keeler contraction. Therefore, there exists an element $\widetilde{p} \in \cap_{i\geq 0} H_i$ with $\widetilde{p} = U_* \widetilde{p}$. Put $\widetilde{z}_k = P_{H_k} f(\widetilde{p})$. Then, $\widetilde{z}_k \to \widetilde{p}$ as $k \to \infty$, (see [22]). This fact and the convergence of $\{x_k\}$ to $\widetilde{p}$ were proved also in [21].

iii) *We prove that $\widetilde{p} \in$ Fix(T).*

First, note that $x_{k+1} \in H_{k+1}$. By definition, $\|v_k - x_{k+1}\| \leq \|x_k - x_{k+1}\| \to 0$. Consequently, $\|v_k - x_k\| \leq 2\|x_k - x_{k+1}\| \to 0$. Now, from (2.1) it follows that $\|x_k - Tx_k\| \leq 2\|v_k - x_k\|/\overline{\epsilon} \to 0$. Therefore, $\widetilde{p} \in$ Fix(T).

iv) $y_k \to J_\gamma^B \widetilde{p}$ *that is a solution of* $0 \in Ax + By$.

From (1.6), we have that $\|z_k - y_k\| = \|v_k - x_k\|/\epsilon_k \leq \|v_k - x_k\|/\underline{\epsilon} \to 0$. On other hand,

$$\{(y_k, x_k - y_k)\} \subset Gph(\gamma B), (x_k, u_k) \overset{w}{\to} (J_\gamma^B \widetilde{p}, \widetilde{p} - J_\gamma^B \widetilde{p}), u_k = x_k - y_k,$$

$$\{(z_k, 2y_k - x_k - z_k)\} \subset Gph(\gamma A), (z_k, w_k) \overset{w}{\to} (J_\gamma^B \widetilde{p}, -\widetilde{p} + J_\gamma^B \widetilde{p}), w_k = 2y_k - x_k - z_k, \|u_k + w_k\| \to 0.$$

By Proposition 1.1, $J^B_\gamma \widetilde{p} \in Zer(A+B)$. This completes the proof.

Theorem 2.2. *Let A, B and $\{c_k\}$ be as in Theorem 2.1. Then, we have the same conclusion for $\{x_k\}$, defined by* (1.7).

Proof. Clearly, H_k and W_k are closed convex subsets of H and $\mathrm{Fix}(T) \subset H_k$ for all $k \geq 0$. We prove that $\mathrm{Fix}(T) \subset W_k$ for every $k \in \mathbb{N}$ and a sequence $\{x_k\}$ is well defined. We have $W_0 = H$, so $\mathrm{Fix}(T) \subset W_0$. We shall prove this fact by mathematical induction. Suppose that $Fix(T) \subset H_n \cap W_n$ for some $n \in \mathbb{N}$. We prove that $\mathrm{Fix}(T) \subset H_{n+1} \cap W_{n+1}$. Since $\mathrm{Fix}(T) \subset H_n \cap W_n$ there exists a unique element $x_k + 1 = P_{H_k \cap W_k} f(x_k)$, and hence, $\langle f(x_k) - x_{k+1}, x_{k+1} - p\rangle \geq 0$ for all $p \in H_n \cap W_n$, which implies that $\langle f(x_k) - x_{k+1}, x_{k+1} - p\rangle \geq 0$ for all $p \in \mathrm{Fix}(T)$. Thus, $\mathrm{Fix}(T) \subset W_{n+1}$. Since $P_{\cap_{i\geq 0}} W_i f$ is a Meir-Keeler contraction in H, there exists an element $\widetilde{p} \in H$ such that $\widetilde{p} \in \cap_{i\geq 0} H_i$ with $\widetilde{p} = P_{\cap_{i\geq 0}} W_i f(\widetilde{p})$. Put $\widetilde{z}_k = P_{H_k} f(\widetilde{p})$. Then $\widetilde{z}_k \to \widetilde{p}$ as $k \to \infty$. The left arguments as in the proof of Theorem 2.1. This completes the proof.

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References

[1] B. Martinet, Regularisation d'inequations variationelles par approximations succivees, Revue Francaise d'Informatiques et de Recherche Operationelle, (1970) 154-159.

[2] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877-898.

[3] O. Guler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim. 29 (1991) 403-419.

[4] G.I. Marchuk, Methods of Numercial Mathesmatic, Springer, New York, 1975.

[5] D. Gaby, Applications of the method of multipliers to variational inequalities, in: M. Fortin and R. Glowinski, eds., *Augmented Lagragian Methods: Applications to the Solution of Bounty Value Problems* (North-Holland, Amsterdam,1983).

[6] C.B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilber spaces, J. Math. Anal. Appl. 72 (1979) 383-390.

[7] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, SIAM J. Control Optim. 29(1) (1991) 119-138.

[8] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone operator Theory in Hibert Spaces, Springer, 2011.

[9] L.P. Lions, Une mesthode iterative de resolution dune inequation variationelle, Israel J. Math. 31 (1978) 204-208.

[10] L.P. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, SIAM J. Numer. Anal. 16 (1979) 964-979.

[11] J. Eckstein, F.F. Svaiter, A family of projective splitting methods for the sum of two monotone operators, Math. Program. Ser. B 111 (2008) 173-199.

[12] B.F. Svaiter, On weak convergence of the Douglas-Rachford method. SIAM J. Control Optim. 49(1) (2011) 280-287.

[13] J. Douglas, H. Rachford, On the numerical solution of heat conduction problem in two and three spaces variables, Trans. Amer. Math. Soc. 82 (1956) 421-439.

[14] H. Zhang, L. Cheng, Projective splitting methods for sum of maximal monotone operators with applications, J. Math. Anal. Appl. 406 (2013) 323-334.

[15] Y. Haugazeau, Sur le Inquations Varialtionnelles et la Minimization de Functionnelles Convexes, in: These de Doctorat, Universit de Paris, France. 1968.

[16] M.V. Solodov, B.F. Svaiter, Forcing strong convergence of proximal point iterations in Hilbert spaces, Math. Program. 87 (2000) 189-202.

[17] K. Nakajo, W. Takahashi, Strong convergence theorem for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372-378.

[18] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorem by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276-286.

[19] K. Aoyama, F. Kohsaka, W. Takahashi, Shrinking projection methods for family of nonexpansive mappings, Nonlinear Analysis, 71 (2009) 1626-1632.

[20] A. Meir, E. Keeler, A theorem on contraction mapping, J. Math. Anal. Appl. 28 (1969) 326-329.

[21] Y. Kimura, K. Nakajo, Viscosity approximations by shrinking projection method in Hilbert spaces, Comput. Math. Appl. 63 (2012) 1400-1408.

[22] M. Tsukada, Convergence of best approximations in a smooth Banach space, J. Approx. Theory, 40 (1984) 301-309

TÓM TẮT

Phương pháp chiếu co rút và cải biên phương pháp tách lai ghép Douglas-Rachford cho toán tử đơn điệu trong không gian Hilbert

Trong bài báo này, để tìm không điểm cho một toán tử đơn điệu trong không gian Hilbert, chúng tôi giới thiệu phương pháp chiếu co rút và phương pháp tách lai ghép Douglas-Rachford. Các phương pháp này được dựa trên phương pháp chiếu co rút của Takahashi cho ánh xạ không giãn, phương pháp lai ghép và phương pháp tách Douglas-Rachford.

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