ON THE EXISTENCE OF SOLUTIONS TO GENERALIZED QUASIVARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT

In this paper, we study the existence theorems of solution for generalized quasivariational inequality problems.

Keywords: *upper and lower C-convex multivalued mappings, upper and lower C-quasiconvex-like multivalued mappings, upper and lower C- continuous multivalued mappings.*

1 Introduction

Throuthout this paper, X is denoted a real Hausdorff locally convex topological vector space; Y be a real topological vector space and let $C \subseteq Y$ be a cone. We put $l(C) = C \cap (-C)$. If $l(C) = \{0\}$, C is said to be pointed. Let Y^* be the topological dual space of Y. We denote by $\langle \xi, y \rangle$ the duality pair between $\xi \in Y^*$ and $y \in Y$. The topological dual cone C' and strict topological dual cone C'^+ of C are defined as

$$C' := \{\xi \in Y' : \langle \xi, c \rangle \geq 0 \text{ for all } c \in C\},$$

$$C'^+ := \{\xi \in Y' \quad \langle \xi, c \rangle > 0 \text{ for all } c \in C \backslash l(C)\}.$$

In this paper, we assume that C be a pointed cone with $C'^+ \neq \emptyset$. Let $D \subseteq X$. be nonempty subset. Given multivalued mappings $S : D \to 2^D$ with nonempty values and $F : D \times D \to 2^Y$ with nonempty compact values. For any $\xi \in C'^+$, we consider the following generalized quasivariational inequality problems:

(P_ξ) Find $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and

$$\max_{z \in F(\bar{x}, \bar{x})} \langle \xi, z \rangle \leq \max_{z \in F(x, \bar{x})} \langle \xi, z \rangle$$

for all $x \in S(\bar{x})$.

In the cases F is a real function on $D \times D$ and $C = \mathbb{R}_+$ then problem (P_ξ) becomes to find $\bar{x} \in D$ such that $x \in S(\bar{x})$ and $F(\bar{x}, \bar{x}) \leq F(x, \bar{x})$ for all $x \in S(\bar{x})$.

This is scalar quasivariational inequality problem studied in [3]. We know that scalar quasivariational inequality problem as generalizations of variational inequalities and optimization problems, including also many other related optimization problems such as fixed point problems, complementarity problems, Nash equilibria problems, minimax problems, etc. The purpose of this article is to establish sufficient conditions for the existence of solutions to problems (P_ξ).

Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted

by

$$\operatorname{dom} F := \{x \in D : F(x) \neq \emptyset\},$$

$$\operatorname{gph} F := \{(x, y) \in D \times Y : y \in F(x)\},$$

respectively. We recall that F is said to be a closed (respectively, open) mapping if $\operatorname{gph} F$ is a closed (respectively, open) subset in the product space $X \times Y$. A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) semicontinuous in $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap V \neq \emptyset$) there exists an open set U of $\bar{x}$ such that $F(x) \subseteq V$ (respectively, $F(x) \cap V \neq \emptyset$) for all $x \in U$.

Definition 1.1. Let Y be a linear space and C a nontrivial convex cone in Y. A nonempty convex subset B of C is called a base for C if each nonzero element $x \in C$ has a unique representation of the form $x = \lambda b$ with $\lambda > 0$ and $b \in B$.

Proposition 1.2 (See [2]). *Let Y be a Hausdorff locally convex space and C is a nontrivial cone in Y. Then C has a base B with $0 \notin \operatorname{cl} B$ if and only if $C'^+ \neq \emptyset$.*

Remark 1.3. If Y is locally convex Hausdorff space, C has a convex weakly* compact base, then $C'^+ \neq \emptyset$.

The following definitions will be used in the sequel.

Definition 1.4. Let $F : D \to 2^Y$ be a multivalued mapping. We say that F is a upper (lower) C-continuous in $\bar{x} \in \operatorname{dom} F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:

$$F(x) \subseteq F(\bar{x}) + V + C$$

$$(F(\bar{x}) \subseteq F(x) + V - C, \text{ respectively})$$

holds for all $x \in U \cap \operatorname{dom} F$.

If F is upper C-continuous, lower C-continuous in any point of $\operatorname{dom} F$, we say that it is upper C-continuous, lower C-continuous on D.

Definition 1.5. Let $F : D \times D \to 2^Y$ be a multivalued mapping. We say that:

(i) F is diagonally upper (lower) C-convex in the first variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$, one have

$$\sum_{j=1}^{n} \alpha_j F(x_j, x) \subseteq F(x, x) + C,$$

$$\text{(respectively, } F(x, x) \subseteq \sum_{j=1}^{n} \alpha_j F(x_j, x) - C).$$

(ii) F is diagonally upper (lower) C-quasiconvex-like in the first variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0 (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$. there exists an index $i \in \{1, ..., n\}$ such that

$$F(x_i, x) \subseteq F(x, x) + C.$$

$$(\text{ respectively, } F(x, x) \subseteq F(x_i, x) - C).$$

To prove the main results, we need the following theorem.

Theorem 1.6. (Fan- Browder, see [1]) *Let D be a nonempty convex compact subset of a topological vector space, $F : D \to 2^D$ be a multivalued map. Suppose that*

(i) *$F(x)$ is a nonempty convex subset of D for each $x \in D$;*

(ii) *$F^{-1}(x)$ is open in D for each $x \in D$. Then there exists $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$.*

2 Existence of solutions for generalized quasivariational inequality problems

In this section we wish to establish an existence criterion for solutions of generalized quasivariational inequality problems. Let $M : D \to 2^D$ be a multivalued mapping. First of all, we prove the following proposition.

Proposition 2.1. *Suppose that D is a nonempty convex compact subset and the multivalued mappings S and M satisfy the following conditions:*

(i) *S has nonempty values, $S^{-1}(x)$ is open for all $x \in D$ and the set $W := \{x \in D : x \in S(x)\}$ is nonempty closed;*

(ii) *for each $t \in D$, the set*

$$B_t := \{x \in D : t \in M(x)\}$$

is closed in D;

(iii) *for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$, there exists an index $j \in \{1, 2, ..., n\}$ such that $t_j \in M(x)$. Then there exists $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and*

$$x \in M(\bar{x}) \text{ for all } x \in S(\bar{x}).$$

Proof. We define the multivalued mapping $G : D \to 2^D$ by

$$G(x) = \{t \in D : t \notin M(x)\}.$$

By (iii) we have $G^{-1}(t) = D \backslash B_t$ is open in D, for all $t \in D$. We show that there exists $\bar{x} \in W$ such that $G(\bar{x}) \cap S(\bar{x}) = \emptyset$. On the contrary, suppose that $G(x) \cap S(x) \neq \emptyset$ for all $x \in W$. Now, we define the multivalued mapping $H : D \to 2^D$ by

$$H(x) = \begin{cases} \mathrm{co}\, G(x) \cap \mathrm{co}\, S(x), & \text{if } x \in W, \\ \mathrm{co}\, S(x), & \text{otherwise.} \end{cases}$$

Then $H(x)$ are nonempty convex for all $x \in D$ and

$$H^{-1}(x') = [(\mathrm{co}\, G)^{-1}(x') \cap (\mathrm{co}\, S)^{-1}(x')] \cup$$

$$[(\mathrm{co}\, S)^{-1}(x') \cap D \backslash W] \text{ is open in } D.$$

By Fan- Browder fixed point theorem, there exists $x^* \in D$ such that $x^* \in H(x^*)$. Hence $x^* \in S(x^*)$ and $x^* \in \mathrm{co}\, G(x^*)$. This implies, there exists $\{t_1, t_2, ..., t_n\} \subseteq G(x^*)$ such that $x^* \in co\{t_1, t_2, ..., t_n\}$. By definition of G, for each $i \in \{1, 2, ..., n\}$, $t_i \notin M(x^*)$. This contradicts with (iii). Hence there exists $\bar{x} \in W$ such that $G(\bar{x}) \cap S(\bar{x}) = \emptyset$. This implies $\bar{x} \in S(\bar{x})$ and

$$x \in M(\bar{x}) \text{ for all } x \in S(\bar{x}).$$

The proof is complete. □

Now we are able to establish sufficient conditions for existence of solutions to (P_ξ).

Theorem 2.2. *Suppose that D is a nonempty convex compact subset and F with nonempty compact values. Assume that there exists $\xi \in C'^+$ such that the following conditions are fulfilled:*

(i) *S has nonempty values, $S^{-1}(x)$ is open for all $x \in D$ and the set $W := \{x \in D : x \in S(x)\}$ is nonempty closed;*

(ii) *for each $t \in D$, the set*

$$\Gamma_\xi^t := \{x \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t,x)} \langle \xi, z \rangle\}$$

is closed in D;

(iii) *for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$, there exists an index $j \in \{1, 2, ..., n\}$ such that*

$$\max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t_j,x)} \langle \xi, z \rangle.$$

Then (P_ξ) has a solution.

Proof. We define the multivalued mapping $M : D \to 2^D$ by

$$M(x) = \{t \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t,x)} \langle \xi, z \rangle\}.$$

For any fixed $t \in D$, the set

$$\begin{aligned} B_t &= \{x \in D : t \in M(x)\} \\ &= \{x \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t,x)} \langle \xi, z \rangle\} \\ &= \Gamma_\xi^t \end{aligned}$$

is closed set in D. Now, let $\{t_1, t_2, ..., t_n\} \subseteq D$ and $x \in \text{co}\{t_1, t_2, ..., t_n\}$. By (iv) we have there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t_j,x)} \langle \xi, z \rangle.$$

This implies $t_j \in M(x)$. Therefore, all the conditions of Proposition 2.1 are satisfied. Applying Proposition 2.1, there exists $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and

$$x \in M(\bar{x}) \text{ for all } x \in S(\bar{x}).$$

This implies $\bar{x} \in S(\bar{x})$ and

$$\max_{z \in F(\bar{x},\bar{x})} \langle \xi, z \rangle \leq \max_{z \in F(x,\bar{x})} \langle \xi, z \rangle \text{ for all } x \in S(\bar{x}).$$

The proof is complete. □

Example 2.3. Consider problem (P_ξ) where $X = Y = \mathbb{R}, C = \mathbb{R}_- := (-\infty, 0], D \quad [0,1], S(x) = [0,1]$ for all $x \in [0,1]$ and the multivalued mapping $F : D \times D \to 2^{\mathbb{R}}$ by

$$F(x,t) = \begin{cases} [0,x], & \text{if } x \leq t, \\ [x,1], & \text{otherwise.} \end{cases}$$

We easily check that $C'^+ = (-\infty, 0)$. Moreover, for each $\xi \in C'^+$ and $x, t \in [0,1]$, we have

$$\begin{aligned} \max_{z \in F(x,x)} \langle \xi, z \rangle &= \max_{z \in [0,x]} \langle \xi, z \rangle = 0, \\ \max_{z \in F(x,t)} \langle \xi, z \rangle &= \begin{cases} 0, & \text{if } x \leq t, \\ \xi x, & \text{if } x > t. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} \Gamma_\xi^t &:= \{x \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t,x)} \langle \xi, z \rangle\} \\ &= [0,t] \text{ is closed in } D. \end{aligned}$$

On the other hand, for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x = \sum_{i=1}^n \alpha_i t_i, \alpha_i \geq 0$ $(i = 1, 2, ..., n)$, $\sum_{i=1}^n \alpha_i = 1$, there exists an index $j \in \{1, 2, ..., n\}$ such that $x \leq t_j$. This implies

$$\max_{z \in F(x,x)} \langle \xi, z \rangle = \max_{z \in F(t_j,x)} \langle \xi, z \rangle = 0.$$

Then the assumptions in Theorem 2.2 are satisfied and $\bar{x} = 1$ is a unique solution of (P_ξ).

Remark 2.4. For each $\xi \in C'^+$, the assumption (ii) in Theorem 2.2 is satisfied provided that: for any $t \in D$, $F(t, .)$ is upper $(-C)$- continuous and the multivalued mapping $G : D \to 2^Y$ defined by $G(x) = F(x,x)$ is lower C- continuous.

Proof. For $\varepsilon > 0$ be arbitrary, since the continuity of ξ, there exists a neighborhood V of the origin in Y such that $\xi(V) \subseteq (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Let $\{x_\alpha\}$ be a net from Γ_ξ^t converging to x_0. Then, we have

$$\max_{z \in F(x_\alpha,x_\alpha)} \langle \xi, z \rangle \leq \max_{z \in F(t,x_\alpha)} \langle \xi, z \rangle \text{ for all } \alpha.$$

On the other hand, since $F(t, .) : D \to 2^Y$ is upper $(-C)$- continuous and the multivalued mapping $G : D \to 2^Y$ defined by

$G(x) = F(x,x)$ is lower C- continuous, there exists an index α_0 such that

$$\begin{aligned} F(t,x_\alpha) &\subseteq F(t,x_0) - C + V, \\ F(x_0,x_0) &\subseteq F(x_\alpha,x_\alpha) - C + V \end{aligned}$$

for all $\alpha \geq \alpha_0$.

It follows that

$$\begin{aligned} \max_{z\in F(t,x_\alpha)} \langle \xi, z\rangle &< \max_{z\in F(t,x_0)} \langle \xi, z\rangle + \frac{\epsilon}{2}, \\ \max_{z\in F(x_0,x_0)} \langle \xi, z\rangle &< \max_{z\in F(x_\alpha,x_\alpha)} \langle \xi, z\rangle + \frac{\epsilon}{2} \end{aligned}$$

for all $\alpha \geq \alpha_0$.

Hence

$$\max_{z\in F(x_0,x_0)} \langle \xi, z\rangle < \max_{z\in F(t,x_0)} \langle \xi, z\rangle + \epsilon.$$

Therefore,

$$\max_{z\in F(x_0,x_0)} \langle \xi, z\rangle \leq \max_{z\in F(t,x_0)} \langle \xi, z\rangle.$$

This shows $x_0 \in \Gamma_\xi^t$. Consequently, Γ_ξ^t is closed. □

Remark 2.5. *For $\xi \in C'^+$, the condition* (iii) *of Theorem 2.2 is satisfied if one of the following conditions is satisfied:*

1. *for each $x \in D$, the set*

$$\Omega_\xi^x := \{t \in D : \max_{z\in F(x,x)} \langle \xi, z\rangle > \max_{z\in F(t,x)} \langle \xi, z\rangle\}$$

is convex.

2. *$F(.,.) : D \times D \to 2^Y$ is diagonally lower C- convex in the first variable.*

3. *F is diagonally lower C- quasiconvex-like in the first variable.*

Proof. Let $\{t_1, t_2, ..., t_n\} \subseteq D$ and $x \in \text{co}\{t_1, t_2, ..., t_n\}$.

1. Assume that for each $j \in \{1, 2, ..., n\}$ we have

$$\max_{z\in F(x,x)} \langle \xi, z\rangle > \max_{z\in F(t_j,x)} \langle \xi, z\rangle.$$

This implies $t_j \in \Omega_\xi^x$ for $j = 1, 2, ..., n$. By Ω_ξ^x is convex set, $x \in \Omega_\xi^x$. This contradicts. Hence there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z\in F(x,x)} \langle \xi, z\rangle \leq \max_{z\in F(t_j,x)} \langle \xi, z\rangle.$$

2. Since $F(.,.)$ is diagonally lower C- convex in the first variable, then

$$F(x,x) \subseteq \sum_{i=1}^n \alpha_i F(t_i, x) - C,$$

where $x = \sum_{i=1}^n \alpha_i t_i, \alpha_i \geq 0$ $(i = 1, 2, ..., n)$, $\sum_{i=1}^n \alpha_i = 1$. This implies

$$\begin{aligned} \max_{z\in F(x,x)} \langle \xi, z\rangle &\leq \max_{z\in \sum_{i=1}^n \alpha_i F(t_i,x)} \langle \xi, z\rangle \\ &\leq \sum_{i=1}^n \alpha_i \max_{z\in F(t_i,x)} \langle \xi, z\rangle \\ &\leq \max_{1\leq i\leq n} \max_{z\in F(t_i,x)} \langle \xi, z\rangle. \end{aligned}$$

Thus, there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z\in F(x,x)} \langle \xi, z\rangle \leq \max_{z\in F(t_j,x)} \langle \xi, z\rangle.$$

3. If F is diagonally lower C- quasiconvex-like in the first variable, there exists an index $j \in \{1, 2, ..., n\}$,

$$F(x,x) \subseteq F(t_j, x) - C.$$

This yields

$$\max_{z\in F(x,x)} \langle \xi, z\rangle \leq \max_{z\in F(t_j,x)} \langle \xi, z\rangle.$$

□

Since Theorem 2.2, Remark 2.4 and Remark 2.5, we have following corollarys :

Corollary 2.6. *Suppose that D is a nonempty convex compact subset and F with nonempty compact values. Assume that there exists $\xi \in C'^{+}$ such that the following conditions are fulfilled:*

(i) *S has nonempty values, $S^{-1}(x)$ is open for all $x \in D$ and the set $W := \{x \in D : x \in S(x)\}$ is nonempty closed;*

(ii) *for each $t \in D$, the set*

$$\Gamma_{\xi}^{t} := \{x \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle \leq \max_{z \in F(t,x)} \langle \xi, z \rangle$$

is closed in D;

(iii) *for each $x \in D$, the set*

$$\Omega_{\xi}^{x} := \{t \in D : \max_{z \in F(x,x)} \langle \xi, z \rangle > \max_{z \in F(t,x)} \langle \xi, z \rangle\}$$

is convex.
Then (P_{ξ}) has a solution.

Corollary 2.7. *Suppose that D is a nonempty convex compact subset and the multivalued mappings S, F satisfy the following conditions:*

(i) *S has nonempty values, $S^{-1}(x)$ is open for all $x \in D$ and the set $W := \{x \in D : x \in S(x)\}$ is nonempty closed;*

(ii) *F has nonempty compact values, for any $x' \in D$, $F(x', .)$ is upper $(-C)$- continuous and the multivalued mapping $G : D \to 2^{Y}$ defined by $G(x) = F(x, x)$ is lower C-continuous;*

(iii) *F is diagonally lower C- convex in the first variable (or, F is diagonally lower C-quasiconvex-like in the first variable).*
Then (P_{ξ}) has a solution, for all $\xi \in C'^{+}$.

Remark 2.8. If $Y = \mathbb{R}, C = \mathbb{R}_{+}$ and $F : D \times D \to \mathbb{R}$ is a single map, then Corollary 2.7 reduces to Corollary 2.4 in [3].

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Summary

Sự tồn tại nghiệm của bài toán bất đẳng thức tựa biến phân suy rộng

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Trong bài báo này, chúng tôi chứng minh một số kết quả cho sự tồn tại nghiệm của bài toán bất đẳng thức tựa biến phân suy rộng.

Key words and phrases: *C-lồi trên và dưới của ánh xạ đa trị, C-giống như tựa lồi trên và dưới của ánh xạ đa trị, C- liên tục trên và dưới của ánh xạ đa trị.*

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