USING METHOD OF LAGRANGE MULTIPLIERS IN THE PROBLEM OF FINDING ABSOLUTE MAXIMUM AND MINIMUM OF FUNTION OF TOW VARIABLES

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SUMMARY

The problem of finding absolute maximum and minimum values of function of two variables on a closed bounded set D have general method. However, when solving some problems, finding critical points on the boundary of D is difficult to put *y= y (x)* or *x= x(y)* to substitute into *f(x, y)* and to make the problem becomes complex. So, this article presents method of Lagrange Multipliers to find critical points on the boundary of D when we solve absolute maximum and minimum problems. Simultaneously, providing some illustrative examples to show the effectiveness of this method, when finding the critical points on the boundary of D in the case of obtaining *y=y(x)* or *x=x(y)* from the boundary equation of *D* and substitute into *f(x,y)* difficultly.

Keywords: *Absolute Maximum, absolute Minimum, function of tow variables, method of Lagrange multipliers, critical point*

INTRODUCTION

Finding absolute Maximum and Minimum problems for the function of tow variables has a general solution. However, some problems are difficult to obtain *y=y(x)* or *x=x(y)* and substitute into *f(x,y)*. Therefor, this article presents method of Lagrange multipliers to find critical points of *f* on the boundary of D in this case.

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Solution method

Problem. Find absolute maximum and minimum of funtion $z= f(x,y)$ on a closed bounded set D.

Solution method [1],[4]

1. Find the values of *f* at the critical points of *f* in D by solving system of equations:

$$\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Rightarrow M_0 \in D \Rightarrow f(M_0)$$

2. Find the extreme values of *f* on the boundary of *D*, assum that $f(M_1), f(M_2),..., f(M_n)$.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value:

$Maxf= Max\{ f(M_0), f(M_1), f(M_2),..., f(M_n)\}$

$Minf= Min\{ f(M_0), f(M_1), f(M_2),..., f(M_n)\}$

However, in step 2, some problems are difficult to obtain *y=y(x)* or *x=x(y)* and substitute into *f(x,y)*. We can use method of Lagrange multiplier to find critical points on the boundary of *D* as follows:

+ Setting the function: $F(x,y)= f(x,y) + \lambda g(x,y)$ ($g(x,y)=0$ is the boundary equation of *D*)

+ Solve system of equations to find critical points on the boundary of *D*:

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ g(x,y) = 0 \end{cases} \Rightarrow M_1, M_2, ..., M_n$$

Examples

Example 1. Find absolute maximum and minimum values of function $z = f(x,y) = x^2 + y^2 + y$ on a closed bounded set D: $x^2 + y^2 \leq 1$

Solution. In this example, we can solve by tow methods

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+ Solve system of equations:

$$\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 0 \\ 2y+1=0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -\frac{1}{2} \end{cases}$$

$$\Rightarrow M_0(0,-\frac{1}{2}) \in D \Rightarrow f(M_0) = \frac{3}{4}$$

+ On the boundary of D $x^2+y^2=1$, we have $x^2 = 1-y^2$ and

$$f(1-y^2,y) = 1-y^2+y^2+y = 1+y,\ y\in[-1,1]$$

This is an increasing funtion of y, so its minimum value is *f(0,-1)=0* and its maximum value is *f(0,1)=2*

We compare these values with the value $f(M_0) = \frac{3}{4}$ at the critical point and conclude that the absolute maximum value is *f(0,1)=2* and the absolute minimum value is *f(0,-1)=0*

In this example, obtaining *y=y(x)* or *x=x(y)* and substitute into *f(x,y)* is very convenient. However, in step 2 we can use method of Lagrange multipliers to find critical points on the boundary as follows:

+ Setting the function $F(x,y)= x^2 + y^2 + y + \lambda(x^2+ y^2-1)$

+ Solve system of equations:

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} 2x+2\lambda x = 0 \\ 2y+1+2\lambda y = 0 \\ x^2+y^2=1 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} M_1(0,1) \\ \lambda = -\frac{1}{2} \end{cases} \\ \begin{cases} M_2(0,-1) \\ \lambda = -\frac{3}{2} \end{cases} \end{array} \right.$$

$f(M_1) = 2,\qquad f((M_2) = 0$

Comparing three values $f(M_0), f(M_1) f(M_2)$, we have absolute maximum value is *f(0, -1)= 2* at $M_1(0, 1)$ and absolute minimum value is *f(0, -1)=0* at $M_2(0, -1)$.

To show the effectiveness of method of Lagrange multipliers**,** we consider the following example.

Example 2. Find absolute maximum and minimum values of funtion $z = f(x,y) = x^2+y^2$ on the closed bounded set D: $\left(x-\sqrt{2}\right)^2 + \left(y-\sqrt{2}\right)^2 = 9$ [2].

Solution

+ Solve system of equations:

$$\begin{cases} z'_y = 0 \\ z'_x = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Rightarrow M_0(0,0) \in D \Rightarrow f(M_0) = 0$$

+ On the boundary of D:

$$\left(x-\sqrt{2}\right)^2 + \left(y-\sqrt{2}\right)^2 = 9$$

We use method of Lagrage multipliersm.

+ Setting the function:

$$F(x,y) = x^2+y^2+\lambda\left[\left(x-\sqrt{2}\right)^2 + \left(y-\sqrt{2}\right)^2 - 9\right]$$

+ Solve system of equations:

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+2\lambda(x-\sqrt{2}) = 0 \\ 2y+2\lambda(y-\sqrt{2}) = 0 \\ (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x-y)(1+\lambda) = 0 \\ 2y+2\lambda(y-\sqrt{2}) = 0 \\ (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} x = y \\ 2y+2\lambda(y-\sqrt{2}) = 0 \\ (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9 \end{cases} \\ \begin{cases} \lambda = -1 \\ 2y+2\lambda(y-\sqrt{2}) = 0 \\ (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9 \end{cases} \end{array} \right.$$

$$\Leftrightarrow \left[\begin{array}{l} M_1\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right), \lambda_1 = -\frac{5}{3} \\ M_2\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \lambda_2 = -\frac{1}{3} \end{array}\right.$$

$f(M_1) = 25,\ f(M_2) = 1$

Comparing three values $f(M_0),\ f(M_1),\ f(M_2)$, we have absolute maximum value is *f(M₁)= 25* at $M_1\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ and absolute minimum value is *f(M₀)=0* at $M_0(0,0)$.

In **Example 2**, we see that when considering the boundary of domain D that obtain y=y(x) or x=x(y) to subsitutec into f (x, y), the problem becomes complex. So in this example, using Lagrange multipliers to find critical points on the boundary of D makes the problem simpler and easier.

Notice that, we only focus critical points on the boundary, so when solving the system of equations above, we can not find the λ. To see the advantages of this method, we consider the following example.

Example 3. Find absolute maximum and minimum values of funtion $z = f(x,y) = 1 - x^2 - y^2$ on the closed bounded set D: $(x-1)^2 + (y-1)^2 \le 1$

Solution

+ Solve system of equations:

$$\begin{cases} z'_y = 0 \\ z'_x = 0 \end{cases} \Leftrightarrow \begin{cases} -2x = 0 \\ -2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Rightarrow M_0(0,0) \Rightarrow f(M_0) = 1$$

+ On the boundary of D:

$$(x-1)^2 + (y-1)^2 = 1$$

We use method of Lagrage multipliers.

+ Setting the function:

$$F(x,y) = 1 - x^2 - y^2 + \lambda\left[(x-1)^2 + (y-1)^2 - 1\right]$$

+ Solve system of equations:

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ (x-1)^2 + (y-1)^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x + 2\lambda(x-1) = 0 \\ -2y + 2\lambda(y-1) = 0 \\ (x-1)^2 + (y-1)^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x-y)(\lambda-1) = 0 \\ -2y + 2\lambda(y-1) = 0 \\ (x-1)^2 + (y-1)^2 = 1 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} x = y \\ -2y + 2\lambda(y-1) = 0 \\ (x-1)^2 + (y-1)^2 = 1 \end{cases} \\ \begin{cases} \lambda = 1 \\ -2y + 2\lambda(y-1) = 0 \\ (x-1)^2 + (y-1)^2 = 1 \end{cases} \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} M_1\left(\frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}}\right) \\ M_2\left(\frac{\sqrt{2}-1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right) \end{array}\right.$$

$f(M_1) = -2 - 2\sqrt{2},\ f(M_2) = -2 + 2\sqrt{2}$

Comparing three values $f(M_0),\ f(M_1),\ f(M_2)$, we have absolute maximum value is *f(M₂)=* $-2+2\sqrt{2}$ at $M_2\left(\frac{\sqrt{2}-1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$ and absolute minimum value is *f(M₁)=* $-2+2\sqrt{2}$ at $M_1\left(\frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}}\right)$.

Finally, we consider the following example to see the great effect of method of Lagrange multipliers.

Example 4. Find absolute maximum and minimum values of funtion $z = f(x,y) = x^2 + y^2 + 4x - 4y$ on the closed bounded set D: $x^2 + y^2 \le 9$ [3, 987].

Solution

+ Solve system of equations:

$$\begin{cases} z'_y = 0 \\ z'_x = 0 \end{cases} \Leftrightarrow \begin{cases} 2x+4=0 \\ 2y-4=0 \end{cases} \Leftrightarrow \begin{cases} x=-2 \\ y=2 \end{cases}$$

$$\Rightarrow M_0(-2,2) \in D \Rightarrow f(M_0) = -8$$

+ On the boundary of D: $x^2 + y^2 = 9$

We use method of Lagrage multipliers

+ Setting the function:

$$F(x,y) = x^2 + y^2 + 4x - 4y + \lambda\left(x^2 + y^2 - 9\right)$$

+ Solve system of equations:

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ x^2 + y^2 = 9 \end{cases} \Leftrightarrow \begin{cases} 2x+4+2\lambda x = 0 \\ 2y-4+2\lambda y = 0 \\ x^2+y^2 = 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x+y)(\lambda+1) = 0 \\ 2y-4+2\lambda y = 0 \\ x^2+y^2 = 9 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} x = -y \\ 2y-4+2\lambda y = 0 \\ x^2+y^2 = 9 \end{cases} \\ \begin{cases} \lambda = -1 \\ 2y-4+2\lambda y = 0 \\ x^2+y^2 = 9 \end{cases} \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} M_1\left(-\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right) \\ M_2\left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right) \end{array}\right.$$

$$f(M_1) = 9 - 12\sqrt{2},\ f(M_2) = 9 + 12\sqrt{2}$$

Comparing three values $f(M_0), f(M_1), f(M_2)$, we have absolute maximum value is *f(M₂)*= $9+12\sqrt{2}$ at $M_2\left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right)$ and absolute minimum value is *f(M₀)*= *-8* at $M_0(-2,2)$.

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TÓM TẮT

SỬ DỤNG PHƯƠNG PHÁP NHÂN TỬ LAGRANG TRONG BÀI TOÁN TÌM GIÁ TRỊ LỚN NHẤT VÀ GIÁ TRỊ NHỎ NHẤT CỦA HÀM HAI BIẾN

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Bài toán tìm giá trị lớn nhất, giá trị nhỏ nhất của hàm hai biến trên tập đóng D đã có phương pháp giải chung. Tuy nhiên, khi giải một số bài toán, việc tìm điểm tới hạn trên biên của D bằng cách rút y= y(x) hoặc x= x(y) để thay vào hàm f(x,y) là rất khó khăn hoặc làm cho bài toán trở nên phức tạp hơn. Vì vậy bài báo này trình bày phương pháp nhân tử Lagrang để tìm điểm tới hạn trên biên của D khi giải bài toán tìm giá trị lớn nhất, giá trị nhỏ nhất. Đồng thời đưa ra một số ví dụ minh họa cho thấy tính hiệu quả của phương pháp này khi tìm điểm tới hạn trên biên của D, mà việc rút y theo x hoặc x theo y để thay vào hàm f(x,y) gặp khó khăn.

Từ khóa: *Giá trị lớn nhất, giá trị nhỏ nhất, hàm hai biến, phương pháp nhân tử Lagrang, điểm tới hạn*

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