AN ALGORITHM TO ESTIMATE THE REGION OF ATTRACTION FOR NONAUTONOMOUS DYNAMICAL SYSTEMS

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ABSTRACT

This paper aims to present an algorithm to find a lower estimate for the region of attraction of a nonautonomous system. This work is an extension for the result presented by Tiep D.V and Hue T.T (2018), in which we mention to the problem for only the case of an autonomous system with an exponentially stable equilibrium point. The approach implemented here is to use a linear programming to construct a continuous, piecewise affine (or CPA for brevity) Lyapunov-like function. From this, the estimate is going to be executed effectively.

Keywords: *region of attraction, nonautonomous system, linear programming, Lyapunov theory, CPA Lyapunov function.*

INTRODUCTION

Constructing a CPA Lyapunov function for a nonlinear dynamical system with the use of linear programming were presented properly in detail by S.F. Hafstein ([1], [3]). In the construction such a function, regions $\mathcal{U},\mathcal{D}$ ($\mathcal{D}\subset\mathcal{U}$) of the state-space containing the origin (which is supposed to be the equilibrium point) are used and $\mathcal{U}\backslash\mathcal{D}$ is partitioned into n-simplices. Then, on this set (called Δ) of such n-simplices, a linear programming problem (abbreviated to LPP) is constructed with the variables are assigned to the values at vertices of Δ of a continuous piecewise affine (abbreviated to CPA) function which by fulfilling the constraints of the LPP becomes a Lyapunov or Lyapunov-like function of the system. Then, a search for a feasible solution for the LPP on Δ is executed. If this search succeeds then we get a Lyapunov-like function if $\mathcal{D}\neq\emptyset$, or a true Lyapunov function if $\mathcal{D}=\emptyset$. Basing on this, an estimate of the region of attraction or an implication for the behavior of the trajectories near the equilibrium will be uncovered.

Concretely, we consider the system

$$\dot{\mathbf{x}}(t)=\mathbf{f}\big(t,\mathbf{x}(t)\big),\mathbf{x}(t)\in\mathbb{R}^{\mathrm{n}},\forall t\geq 0. \qquad (0.1)$$

Assume that $\mathcal{U}$ is a domain of $\mathbb{R}^n$ and $\mathbf{x}^*=\mathbf{0}\in\mathcal{U}$ is an equilibrium, and that

$$\mathbf{f}=(f_1,f_2,\dots,f_n):\mathbb{R}_{\geq 0}\times\mathcal{U}\to\mathbb{R}^{\mathrm{n}} \qquad (0.2)$$

is locally Lipschitz. For each $t_0\geq 0$, and each $\boldsymbol{\xi}\in\mathcal{U}$, assume that $t\longmapsto\phi(t,t_0,\boldsymbol{\xi})$ is the solution of (1.1) such that $\phi(t_0,t_0,\boldsymbol{\xi})=\boldsymbol{\xi}$. Then, the region of attraction of the equilibrium at the origin of the system (1.1) with respect to t_0 is defined by

$\mathcal{R}^{t_0}\coloneqq\{\boldsymbol{\xi}\in\mathcal{U}:\lim_{t\to\infty}sup\,\phi(t,t_0,\boldsymbol{\xi})=0\}$.

The region of attraction of the equilibrium at the origin is defined by

$$\mathcal{R}\coloneqq\bigcap_{t_0\geq 0}\mathcal{R}^{t_0}$$

$$=\{\boldsymbol{\xi}\in\mathcal{U}:\lim_{t\to\infty}\phi(t,t_0,\boldsymbol{\xi})=0\,,\forall t_0\geq 0\}.$$

Let $0\leq T'<T''$ be constants and $\boldsymbol{PS}:\mathbb{R}^n\to\mathbb{R}^n$ be a piecewise scaling function. Let $\mathcal{N}\subset\mathcal{U}$ be a set such that the interior of

$$\mathcal{M}\coloneqq\bigcup_{\boldsymbol{z}\in\mathbb{Z}^n,\boldsymbol{PS}(\boldsymbol{z}+[0,1]^n)\subset\mathcal{N}}\boldsymbol{PS}(\boldsymbol{z}+[0,1]^n)$$

is the connected set containing the origin. Let $\mathcal{D}\coloneqq\boldsymbol{PS}\left(\left(d_1,\hat{d}_1\right)\times\left(d_2,\hat{d}_2\right)\times\ldots\times\left(d_n,\hat{d}_n\right)\right)$ be the set of which closure is contained in the interior of $\mathcal{M}$, and either $\mathcal{D}=\emptyset$, or $d_i\leq -1$ and $\hat{d}_i\geq 1,\forall i=1,2,\dots,n$. Let $\boldsymbol{t}=(t_0,t_1,\dots,t_M)$ be a vector such that

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$$T' = t_0 < t_1 < \ldots < t_M = T''.$$

Assume that $\mathbf{f}$ has all second order partial derivatives which are continuous and bounded on $[T',T''] \times (\mathcal{M}\backslash\mathcal{D})$. Define the piecewise scaling function $\widetilde{\boldsymbol{PS}}: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$ by

$$\widetilde{\boldsymbol{PS}}(i,\mathbf{x}) = \big(t_i, \boldsymbol{PS}(\mathbf{x})\big), \forall i = 1,2,\ldots,n. \quad (0.3)$$

Define a seminorm $\|\cdot\|_*$ on $\mathbb{R}\times\mathbb{R}^n$ through an arbitrary norm $\|\cdot\|$ of $\mathbb{R}^n$ by $\|(x_0,\mathbf{x})\|_* := \|\mathbf{x}\|, \forall (x_0,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^n$. Define the set $\mathcal{G}$ as $\{\tilde{\mathbf{x}} \in \mathbb{R}\times\mathbb{R}^n | \tilde{\mathbf{x}} \in \widetilde{\boldsymbol{PS}}(\mathbb{Z}\times\mathbb{Z}^n) \cap \big([T',T'']\times(\mathcal{M}\backslash\mathcal{D})\big)\}$, and $\mathcal{X} := \{\|\mathbf{x}\| \mid \mathbf{x} \in \boldsymbol{PS}(\mathbb{Z}^n) \cap \mathcal{M}\}$. Define for each permutation σ of $\{0,1,\ldots,n\}$ a vector $\mathbf{x}_i^\sigma := \sum_{j=i}^n e_{\sigma(i)}, \forall i = 0,1,\ldots,n+1$. Let $\mathcal{Z}$ be the set of all pairs $(\boldsymbol{z},\mathcal{J})$, for each $\boldsymbol{z} \in \mathbb{Z}_{\geq 0}^{n+1}$ and each $\mathcal{J} \subset \{1,2,\ldots,n\}$, such that $\widetilde{\boldsymbol{PS}}\left(\widetilde{\boldsymbol{R}}^{\mathcal{J}}(\boldsymbol{z} + [0,1]^{n+1})\right)$ is contained in $[T',T'']\times(\mathcal{M}\backslash\mathcal{D})$, where

$\widetilde{\boldsymbol{R}}^{\mathcal{J}}(t,x_1,x_2,\ldots,x_n) :=$
$\left(t,(-1)^{\chi_{\mathcal{J}}(1)}x_1,(-1)^{\chi_{\mathcal{J}}(2)}x_2,\ldots,(-1)^{\chi_{\mathcal{J}}(n)}x_n\right)$

and $\chi_{\mathcal{J}}$ is the characteristic function of $\mathcal{J}$. For each $(\boldsymbol{z},\mathcal{J}) \in \mathcal{Z}$, set $\mathbf{y}_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})} := \widetilde{\boldsymbol{PS}}\left(\widetilde{\boldsymbol{R}}^{\mathcal{J}}(\boldsymbol{z} + \mathbf{x}_i^\sigma)\right)$. Let $\mathcal{Y} := \left\{\left(\mathbf{y}_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})}, \mathbf{y}_{\sigma,i+1}^{(\boldsymbol{z},\mathcal{J})}\right) \middle| \sigma, (\boldsymbol{z},\mathcal{J}) \in \mathcal{Z}, i = 0,\ldots,n\right\}$ be the set of every pair neighboring grid points in $\mathcal{G}$. Moreover, $\forall (\boldsymbol{z},\mathcal{J}) \in \mathcal{Z}, \forall r,s = 0,1,\ldots,n$, set $B_{r,s}^{(\boldsymbol{z},\mathcal{J})}$ to be a bound of $\frac{\partial^2 \mathbf{f}}{\partial r \partial s}$ on the set $\widetilde{\boldsymbol{PS}}\left(\widetilde{\boldsymbol{R}}^{\mathcal{J}}(\boldsymbol{z} + [0,1]^{n+1})\right)$. For each σ, define $A_{\sigma,r,s}^{(\boldsymbol{z},\mathcal{J})}$ to be $\left|\mathbf{e}_r \cdot \left(\mathbf{y}_{\sigma,s}^{(\boldsymbol{z},\mathcal{J})} - \mathbf{y}_{\sigma,s+1}^{(\boldsymbol{z},\mathcal{J})}\right)\right|$, where $\mathbf{e}_r$ is the r-th vector in the standard basis of $\mathbb{R}^{n+1}$. Set

$E_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})} := \frac{1}{2}\sum_{r,s=0}^n B_{r,s}^{(\boldsymbol{z},\mathcal{J})} A_{\sigma,r,i}^{(\boldsymbol{z},\mathcal{J})} \left(A_{\sigma,r,i}^{(\boldsymbol{z},\mathcal{J})} + A_{\sigma,r,0}^{(\boldsymbol{z},\mathcal{J})}\right)$.

2. LINEAR PROGRAMMING PROBLEM

The variables of the LPP are $\Pi, \Psi[y], \Gamma[y]$, and $V[\tilde{\mathbf{x}}], C[\tilde{\mathbf{x}},\tilde{\mathbf{y}}]$, $\forall y \in \mathcal{X}, \forall \tilde{\mathbf{x}} \in \mathcal{G}, \forall (\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathcal{Y}$.

The linear constraints of the LPP are:

(LC1) Let $\mathcal{X} = \{y_0,y_1,\ldots,y_K\}$ be numbered in an increasing order. For an arbitrary constant $\varepsilon > 0$, we require that

$\Psi[y_0] = \Gamma[y_0], \varepsilon y_1 \leq \Psi[y_1], \varepsilon y_1 \leq \Gamma[y_1]$,

and that $\forall i = 1,2,\ldots,K-1$,

$$\frac{\Psi[y_i] - \Psi[y_{i-1}]}{y_i - y_{i-1}} \leq \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i}, \quad (0.4)$$

$$\frac{\Gamma[y_i] - \Gamma[y_{i-1}]}{y_i - y_{i-1}} \leq \frac{\Gamma[y_{i+1}] - \Gamma[y_i]}{y_{i+1} - y_i}. \quad (0.5)$$

(LC2) $\forall \tilde{\mathbf{x}} \in \mathcal{G}$: $\Psi[\|\tilde{\mathbf{x}}\|_*] \leq V[\tilde{\mathbf{x}}]$.

If $\mathcal{D} = \emptyset$: $V[\tilde{\mathbf{x}}] = 0$ whenever $\|\tilde{\mathbf{x}}\|_*$.

If $\mathcal{D} \neq \emptyset$, given an arbitrary $\delta > 0$,

$$V[\tilde{\mathbf{x}}] \leq \Psi\left[x_{min,\partial\mathcal{M}}\right] - \delta,$$

for all $\tilde{\mathbf{x}} = (t,\mathbf{x})$ having $\mathbf{x} \in \boldsymbol{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$, where

$x_{min,\partial\mathcal{M}} := \min\{\|\mathbf{x}\| \mid \mathbf{x} \in \boldsymbol{PS}(\mathbb{Z}^n) \cap \partial\mathcal{M}\}$.

Moreover, $\forall i = 1,2,\ldots,n$, and $j = 0,1,\ldots,M$:

$V\left[t_j\mathbf{e_0} + \boldsymbol{PS}(d_i\mathbf{e}_i)\right] \leq -\Pi\boldsymbol{PS}(d_i\mathbf{e}_i)\cdot\mathbf{e}_i$,

$V\left[t_j\mathbf{e_0} + \boldsymbol{PS}(\hat{d}_i\mathbf{e}_i)\right] \leq \Pi\boldsymbol{PS}(\hat{d}_i\mathbf{e}_i)\cdot\mathbf{e}_i$.

(LC3) $\forall (\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathcal{Y}$:

$$-C[\tilde{\mathbf{x}},\tilde{\mathbf{y}}]\,\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty \leq V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}] \leq C[\tilde{\mathbf{x}},\tilde{\mathbf{y}}]\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty \leq \Pi\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty.$$

(LC4) $\forall (\boldsymbol{z},\mathcal{J}) \in \mathcal{Z}, \forall\sigma, \forall i = 0,1,\ldots,n+1$:

$$\sum_{j=0}^n \left(\frac{V\left[\mathbf{y}_{\sigma,j}^{(\boldsymbol{z},\mathcal{J})}\right] - V\left[\mathbf{y}_{\sigma,j+1}^{(\boldsymbol{z},\mathcal{J})}\right]}{\mathbf{e}_{\sigma(j)}\cdot\left(\mathbf{y}_{\sigma,j}^{(\boldsymbol{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\boldsymbol{z},\mathcal{J})}\right)} f_{\sigma(j)}\left(\mathbf{y}_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})}\right) + E_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})} C\left[\mathbf{y}_{\sigma,j}^{(\boldsymbol{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\boldsymbol{z},\mathcal{J})}\right] \right) \leq -\Gamma\left[\left\|\mathbf{y}_{\sigma,i}^{(\boldsymbol{z},\mathcal{J})}\right\|_*\right]. \quad (0.6)$$

Here, f_0 is the constant function 1, defining on $\mathbb{R}_{\geq 0} \times \mathcal{U}$.

The objective function is not needed.

This LPP for the system (1.1) is denoted by $LP(\mathcal{N}, \boldsymbol{PS}, \boldsymbol{t}, \mathcal{D}, \|\cdot\|)$. We have translated the problem of constructing a Lyapunov function into an LPP. Then, for the LPP, there exists an algorithm to search for a feasible solution, the simplex algorithm.

3. PARAMETERIZE A CPA LYAPUNOV FUNCTION FORM A FEASIBLE SOLUTION OF THE LPP

Assuming that the LPP has a feasible solution for variables $\Pi, \Psi[y], \Gamma[y]$, and $V[\tilde{\mathbf{x}}], C[\tilde{\mathbf{x}}, \tilde{\mathbf{y}}]$, for $\forall y \in \mathcal{X}, \forall \tilde{\mathbf{x}} \in \mathcal{G}, \forall (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathcal{Y}$. Let index in an increasing order the set $\mathcal{X} = \{y_0, y_1, \ldots, y_K\}$. Define CPA functions ψ, γ from $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ by: $\psi(0) \coloneqq 0, \gamma(0) \coloneqq 0$, and $\forall i = 0,1, \ldots, K-1$, set

$$\psi(y) \coloneqq \Psi[y_i] + \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i}(y - y_i),$$

$$\gamma(y) \coloneqq \Gamma[y_i] + \frac{\Gamma[y_{i+1}] - \Gamma[y_i]}{y_{i+1} - y_i}(y - y_i),$$

for every $y \in [y_i, y_{i+1}]$, and set

$$\psi(y) \coloneqq \Psi[y_{K-1}] + \frac{\Psi[y_K] - \Psi[y_{K-1}]}{y_K - y_{K-1}}(y - y_{K-1}),$$

$$\gamma(y) \coloneqq \Gamma[y_{K-1}] + \frac{\Gamma[y_K] - \Gamma[y_{K-1}]}{y_K - y_{K-1}}(y - y_{K-1}),$$

for every $y \in (y_K, \infty)$.

Define a CPA function on $\widetilde{\boldsymbol{PS}}^{-1}([T', T''] \times (\mathcal{M} \backslash \mathcal{D}))$ by $W(t, \mathbf{x}) \coloneqq V[t, \mathbf{x}], \forall (t, \mathbf{x}) \in \boldsymbol{\mathcal{G}}$.

Theorem 1. ψ, γ are convex $\mathcal{K}$ functions. For all $(t, \mathbf{x}) \in [T', T''] \times (\mathcal{M} \backslash \mathcal{D})$, we have

$$\psi(\|\mathbf{x}\|) \leq W(t, \mathbf{x}).$$

If $\mathcal{D} = \emptyset$, then $\psi(0) = W(t, \mathbf{0}) = 0$, for all $t \in [T', T'']$. If $\mathcal{D} \neq \emptyset$, then

$$\min_{\substack{\mathbf{x} \in \partial \mathcal{D} \\ t \in [T', T'']}} W(t, \mathbf{x}) \leq \max_{\substack{\mathbf{x} \in \partial \mathcal{M} \\ t \in [T', T'']}} W(t, \mathbf{x}) - \delta.$$

Assume that ϕ is the solution of the system (1.1) satisfying that $\phi(t_0, t_0, \boldsymbol{\xi}) = \boldsymbol{\xi} \in \mathcal{U}$. Then, $\forall (t, \phi(t, t_0, \boldsymbol{\xi}))$ in the interior of the set $[T', T''] \times (\mathcal{M} \backslash \mathcal{D})$, we have

$$\limsup_{h \to 0^+} \frac{W(t+h, \phi(t+h, t_0, \boldsymbol{\xi})) - W(t, \phi(t, t_0, \boldsymbol{\xi}))}{h} \leq -\gamma(\|\phi(t, t_0, \boldsymbol{\xi})\|). \quad (3.1)$$

Proof. Refer to [3].

Since Theorem 1, we see that in the case $\mathcal{D} = \emptyset$, W is a true CPA Lyapunov function for (1.1). The following result suggests us a nice approach to find an estimate of the region of attraction $\mathcal{R}$, which is the main contribution of this paper.

Theorem 2. Let the norm $\|\cdot\|$ in the LPP be a k-norm ($1 \leq \mathrm{k} \leq \infty$). Define the set Ω by

$\Omega \coloneqq \{0\}$ if $\mathcal{D} = \emptyset$, and $\Omega \coloneqq \mathcal{D} \cup \Big\{\mathbf{x} \in \mathcal{M} \backslash \mathcal{D} \mid \max_{t \in [T', T'']} W(t, \mathbf{x}) \leq \max_{t \in [T', T''], \mathbf{y} \in \partial \mathcal{D}} W(t, \mathbf{y})\Big\}$,

if $\mathcal{D} \neq \emptyset$, and the set $\mathcal{A}$ by

$$\mathcal{A} \coloneqq \Big\{\mathbf{x} \in \mathcal{M} \backslash \mathcal{D} \mid \max_{t \in [T', T'']} W(t, \mathbf{x}) < \max_{\substack{t \in [T', T''], \\ \mathbf{y} \in \partial \mathcal{M}}} W(t, \mathbf{y})\Big\}.$$

Set $E_q \coloneqq \|\sum_{i=1}^{n} \mathbf{e}_i\|_q$, where $q \coloneqq \frac{k}{k-1}$ if $1 < k < \infty$, $q \coloneqq 1$ if $k = \infty$, and $q \coloneqq \infty$ if $k = 1$.

Then, we have

(i) If $\exists t \in [T', T''], \exists t_0 \geq 0$, and $\exists \boldsymbol{\xi} \in \mathcal{U}$ such that $\phi(t, t_0, \boldsymbol{\xi}) \in \Omega$, then $\phi(s, t_0, \boldsymbol{\xi}) \in \Omega$ for all $s \in [t, T'']$.

(ii) If $\exists t \in [T', T''], \exists t_0 \geq 0$, and $\exists \boldsymbol{\xi} \in \mathcal{U}$, such that $\phi(t, t_0, \boldsymbol{\xi}) \in \mathcal{M} \backslash \mathcal{D}$, then for each $s \in [t, T'']$ fulfilling that $\phi(s_0, t_0, \boldsymbol{\xi}) \in \mathcal{M} \backslash \mathcal{D}$ for all $t \leq s_0 \leq s$, we have

$$W(s, \phi(s, t_0, \boldsymbol{\xi})) \leq W(t, \phi(t, t_0, \boldsymbol{\xi})) \exp\left(-\Pi \frac{s-t}{\varepsilon E_q}\right). \quad (3.2)$$

(iii) If $\mathcal{D} = \emptyset$, then W is a Lyapunov function for the system (1.1), the equilibrium $\mathbf{x}^* = \mathbf{0}$ is uniformly asymptotically stable, and $\mathcal{A}$ is a subset of the region of attraction $\mathcal{R}$. Moreover, the solution satisfies (3.2) for all $s \in [t, T'']$ if $\phi(t, t_0, \boldsymbol{\xi}) \in \mathcal{A}$ for some $t \in [T', T''], t_0 \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$.

If $\mathcal{D} \neq \emptyset$ and $\phi(t, t_0, \boldsymbol{\xi}) \in \mathcal{A} \backslash \Omega$, for some $t \in [T', T''], t_0 \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, then $\exists T^* \in (t, T'']$ such that (3.2) fulfils $\forall s \in [t, T^*]$, $\phi(T^*, t_0, \boldsymbol{\xi}) \in \partial \mathcal{D}$, and $\phi(s, t_0, \boldsymbol{\xi}) \in \Omega$ for all $s \in [T^*, T'']$.

Proof. Refer to [1], [3].

Note that in Theorem 2, if $\mathcal{D} = \emptyset$, the set $\mathcal{A}$ is an estimate of the region of attraction $\mathcal{R}$.

Enlarging $\mathcal{A}$ as much as possible means getting the best estimate of $\mathcal{R}$. Therefore, an algorithm is naturally arise to look for such an estimate. This is the subject of the next section.

In the case $\mathcal{D} \neq \emptyset$, however, having no feasible solution for the LPP does not mean the nonexistence of the region of attraction, and therefore, it is hopeless to find any its estimate. The only information extracted from this fact is that the region $\mathcal{M}\backslash\mathcal{D}$ on which a feasible solution of the LPP is searched is unsuited or it simply means that the partition performed on that region is not good enough, and should be replaced by a new one as long as the hope for the search of a feasible solution is not ended. The basis of such an endless hope is stated in the following theorem.

Theorem 3. (Constructive converse theorem)

Assuming that $[-a, a]^n \subset \mathcal{R}$, the region of attraction for some $a > 0$, and that $\mathbf{f}$ is Lipschitz, that is there exists a constant $L > 0$ such that $\forall s, t \in \mathbb{R}_{\geq 0}$, and $\forall \mathbf{x}, \mathbf{y} \in [-a, a]^n$,

$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(s, \mathbf{y})\| \leq L(|s - t| + \|\mathbf{x} - \mathbf{y}\|)$.

Assume further that either $\mathbf{x}^* = \mathbf{0}$ is a uniformly asymptotically stable equilibrium point or there exists a Lyapunov function $W \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times [-a, a]^n \backslash \{\mathbf{0}\})$. Then, for every constants $0 \leq T' < T'' \leq \infty$, and for every neighborhood $\mathfrak{D} \subset [-a, a]^n$ of the origin, maybe arbitrarily small, it is possible to parameterize a Lyapunov function

$$W: [T', T''] \times ([-a, a]^n \backslash \mathfrak{D}) \longrightarrow \mathbb{R}.$$

Strategy of the proof. The idea to prove Theorem 3 is as follows: Firstly, choose a positive integer m such that

$$\mathcal{D} := \left(-2^{k-m}a, 2^{k-m}a\right)^n \subset \mathfrak{D},$$

for some integer $1 \leq k < m$. Define the piecewise scaling function $\boldsymbol{PS}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\boldsymbol{PS}(j_1, j_2, \dots, j_n) := a2^{-m}(j_1, j_2, \dots, j_n) \quad (3.3)$$

for all $(j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$, and a vector $\boldsymbol{t} := (t_0, t_1, \dots, t_{2^m})$, where $t_j := T' + j2^{-m}(T'' - T')$ for all $j = 0,1, \dots, 2^m$. It is sufficient to prove that for an arbitrary norm $\|\cdot\|$ of $\mathbb{R}^n$, the LPP $LP([-a, a]^n, \boldsymbol{PS}, \boldsymbol{t}, \mathcal{D}, \|\cdot\|)$ has a feasible solution, whenever m is large enough. However, basing on the non-constructive converse Lyapunov theorem under the condition of a uniformly asymptotically stable equilibrium point for the system (1.1), we can be sure that there exists a way to assign the appropriate values to the variables of the LPP (even we merely know this existence but an appropriate choice for these parameters is determined by the simplex algorithm). For the derivation of these parameters, refer to [1].

4. ALGORITHM TO ESTIMATE THE REGION OF ATTRACTION

In the case when there exists a Lyapunov function $W(t, \mathbf{x})$ in the $\mathcal{C}^2(\mathbb{R}_{\geq 0} \times (\mathcal{O}\backslash\{\mathbf{0}\}))$, for a neighborhood $\mathcal{O}$ of the origin of $\mathbb{R}^n$, or especially, when that origin is a uniformly asymptotically equilibrium point, the following algorithm secures an estimate of the region of attraction.

Algorithm. Let $T' > 0$ be an arbitrary constant. Consider an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. Assume that $\mathbf{f}$ possess all bounded second order partial derivatives on $[0, T'] \times \Omega$ for each compact subset Ω of $\mathbb{R}^n$. Take an integer $N_\propto > 0$ as the limit level, to which we might expect, of the repetition, for searching the prior estimate of $\mathcal{R}$.

Step 1. Initiate a region $[-a, a]^n \subset \mathcal{U}$ by taking a positive number a. Take an arbitrary neighborhood $\mathfrak{D} \subset [-a, a]^n$ of the origin and a constant B such that

$$B \geq \max_{\substack{r,s=0,1,\dots,n \\ [0,T'] \times [-a,a]^n}} \left\| \frac{\partial^2 \mathbf{f}}{\partial \tilde{x}_r \, \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right\|. \quad (4.1)$$

Initiate the integers $N := 0$, and assign to m the smallest positive integer such that

$$\mathcal{D} := (-a2^{-m}, a2^{-m})^n \subset \mathfrak{D}.$$

Step 2a. Define the piecewise scaling function $\boldsymbol{PS}: \mathbb{R}^n \to \mathbb{R}^n$ by: for all $(j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$,

$$\boldsymbol{PS}(j_1, j_2, \dots, j_n) := a2^{-m}(j_1, j_2, \dots, j_n), \quad (4.2)$$

and the vector $\boldsymbol{t} = (t_0, t_1, \dots, t_{2^m})$ by

$$t_j := j2^{-m}T', \forall j = 0,1, \dots, 2^m.$$

Step 2b. For each $k = 0,1, \dots, N$, check that whether the linear programming problems

$$LP([-a,a]^n, \boldsymbol{PS}, \boldsymbol{t}, \mathcal{D})$$

has a feasible solution or not. If one of the LPP has a feasible solution, then go to step 2d. If there is no LPP possessing a feasible solution, then set $m := m+1, N := N+1$ and go back to step 2a if $N < N_\propto$. Otherwise, if $N \geq N_\propto$, move to step 2c.

Step 2c. Decrease the size of the hyper-box $[-a,a]^n \subset \mathcal{U}$ by setting $a := 2^{-1}a$ and go back to step 1.

Step 2d. Use the found feasible solution to parameterize a CPA Lyapunov function for the system (1.1).

Step 3. Use the constructed CPA Lyapunov function to secure an estimate Ω_c of the region of attraction $\mathcal{R}$, where

$$\mathfrak{B}_\alpha := \left\{ \max_{\substack{t\in[0,T'] \\ \mathbf{x}\in[-a,a]^n\setminus\mathcal{D}}} W(t,\mathbf{x}) < \alpha \right\}, \quad (4.3)$$

$$c := \sup\{\alpha > 0 : \mathfrak{B}_\alpha \subset [-a,a]^n\}, (4.4)$$

$$\Omega_c := \mathcal{D} \cup \{\mathbf{x} \in [-a,a]^n\setminus\mathcal{D} : \max_{t\in[0,T']} W(t,\mathbf{x}) \leq c\}. (4.5)$$

Theorem 4. The algorithm always succeeds in finding an estimate of the region of attraction for the system (1.1), whenever the system fulfils the hypotheses preceding the algorithm.

Proof. This is a straightforward consequence of Theorem 3.

Remark. Let a, k and m be the number with which we obtain a feasible solution for the corresponding LPP. Define the set

$$\mho := \mathcal{D} \cup \{\mathbf{x} \in [-a,a]^n\setminus\mathcal{D} : \max_{t\in[0,T']} W(t,\mathbf{x}) \leq \epsilon\}, (4.6)$$

where $\epsilon := \max_{\substack{t\in[0,T'] \\ \mathbf{x}\in\partial\mathcal{N}}} W(t,\mathbf{x}) > 0$, and

$$\mathcal{D} := \left(-a2^{k-m}, a2^{k-m}\right)^n. \quad (4.7)$$

Then every trajectory starting inside Ω_c will be attracted to $\mho$, and reaches the boundary $\partial\mho$ in a finite period of time, and will be captured in here forever.

5. EXAMPLE

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x})$, where

$$\mathbf{f}(t,\mathbf{x}) = \mathbf{f}(t,x,y) = \begin{bmatrix} -2x + y\cos t \\ x\cos t - 2y \end{bmatrix}$$

This is a nonautonomous linear system. The transition matrix of the system is

$$\Phi(t,t_0) = e^{-2(t-t_0)} \begin{bmatrix} e^{\sin t - \sin t_0} & -e^{-\sin t + \sin t_0} \\ e^{\sin t - \sin t_0} & e^{-\sin t + \sin t_0} \end{bmatrix},$$

satisfying that $\|\Phi(t,t_0)\| \leq Ke^{-2(t-t_0)}$, for some constant $K > 0$. Therefore, the origin is a uniformly asymptotically stable equilibrium point, (cf. [2]). For each $(\boldsymbol{z}, \mathcal{J}) \in \boldsymbol{\mathcal{Z}}$, we set

$$x_{(\boldsymbol{z},\mathcal{J})} := \left|\mathbf{e}_1 \cdot \widetilde{\boldsymbol{PS}}\left(\widetilde{\boldsymbol{R}}^{\mathcal{J}}(\boldsymbol{z} + \mathbf{e}_1)\right)\right|,$$

$$\text{and } y_{(\boldsymbol{z},\mathcal{J})} := \left|\mathbf{e}_2 \cdot \widetilde{\boldsymbol{PS}}\left(\widetilde{\boldsymbol{R}}^{\mathcal{J}}(\boldsymbol{z} + \mathbf{e}_2)\right)\right|.$$

Take $B_{0,0}^{(\boldsymbol{z},\mathcal{J})} := \max\{x_{(\boldsymbol{z},\mathcal{J})}, y_{(\boldsymbol{z},\mathcal{J})}\}$, and

$B_{2,2}^{(\boldsymbol{z},\mathcal{J})} := 0, B_{1,1}^{(\boldsymbol{z},\mathcal{J})} := 0, B_{1,0}^{(\boldsymbol{z},\mathcal{J})} := B_{0,1}^{(\boldsymbol{z},\mathcal{J})} := 1,$

$B_{1,2}^{(\boldsymbol{z},\mathcal{J})} := B_{2,1}^{(\boldsymbol{z},\mathcal{J})} := 0, B_{2,0}^{(\boldsymbol{z},\mathcal{J})} := B_{0,2}^{(\boldsymbol{z},\mathcal{J})} := 1.$

Take the domain $\mathcal{N}$, and $\mathcal{D}$ in the introduction section as $\mathcal{N} := (-0.55, 0.55)^2, \mathcal{D} := (-0.11, 0.11)^2$.

Define the piecewise scaling function $\boldsymbol{PS}$ by

$$\boldsymbol{PS}(\mathbf{x}) := \sum_{i=1}^{n} sign(x_i) P(|x_i|)\mathbf{e}_i,$$

for all $\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{e}_i$, where P is a continuous piecewise affine function and that

$$P: [0,5] \longrightarrow [0, 0.55],$$

$P(j) = 0.11 \times j, \forall j = 0,1,\ldots,5$. Take the vector $\boldsymbol{t} = (0, 0.3125, 0.75, 1.3125, 2)$.

The CPA Lyapunov function $W(t,x,y)$ parameterized from a feasible solution is sketched for the fix time-value $t = 2$ in figure 1. Here, the CPA Lyapunov function

$W: [0,2] \times ([-0.55,0.55]^2 \backslash (-0.11,0.11))^2 \to \mathbb{R}$,

secures an estimate $\Omega_{0.55}$ (cf. figure 2) of the region of attraction $\mathcal{R}$, where

$\Omega_{0.55} := \{\mathbf{x} \in [-0.55,0.55]^n : \max_{t\in[0,2]} W(t,\mathbf{x}) \leq 0.55\}$.

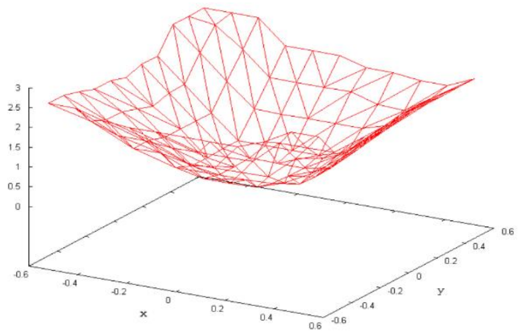

Figure 1. *The graph of the function* $(x,y) \mapsto W(2,x,y)$.

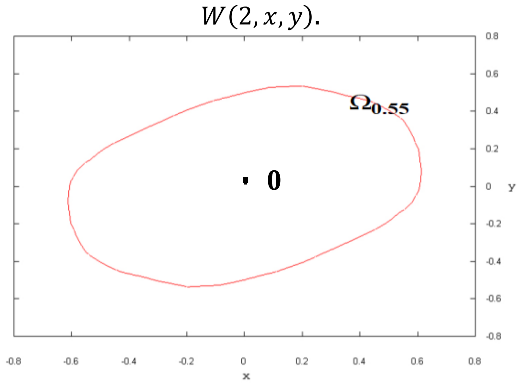

Figure 2. *The estimate* $\Omega_{0.55}$ *of the region of attraction* $\mathcal{R}$.

6. SUMMARY

The algorithm to find a lower estimate for the region of attraction for the case of a nonautonomous system is an extension of one for that of an autonomous system presented in [4]. The strategies are quite similar except for some adjustments to treat the time-varying case by considering the time variable t as an extra state variable x_0 to translate the original system into an autonomous one. Then the algorithm for an autonomous system is applied to the obtained system.

The algorithm always succeeds if the system possesses a uniformly asymptotically stable equilibrium point at the origin. For an autonomous system, a region $\mathcal{D}$ of $\mathbb{R}^n$, which will be excluded from the domain of constructed CPA Lyapunov function, can be ignored if the origin an exponentially stable equilibrium point as presented in [4]. It is not difficult to see that this is also the case for a nonautonomous system possessing a uniformly exponentially stable equilibrium point at the origin. For showing this, we need to modify a little the above algorithm with an extra step of parameterizing the CPA Lyapunov function in a small neighborhood of the origin to cover up the hole $\mathcal{D}$. This method is similar to the algorithm presented in [4].

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TÓM TẮT

MỘT THUẬT TOÁN ƯỚC LƯỢNG MIỀN HẤP DẪN CỦA CÁC HỆ ĐỘNG LỰC PHI Ô-TÔ-NÔM

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Bài báo này nhằm mục đích giới thiệu một thuật toán tìm ước lượng dưới của miền hấp dẫn cho một hệ động lực phi ô-tô-nôm. Kết quả này là một sự mở rộng của kết quả đạt được ở bài báo đưa ra bởi các tác giả Tiệp và Huê (2018), ở đó, bài toán được đặt ra cho hệ ô-tô-nôm với gốc tọa độ là một điểm cân bằng ổn định dạng mũ. Phương pháp tiếp cận được tiến hành ở đây là sử dụng một bài toán quy hoạch tuyến tính để xây dựng một hàm kiểu Lyapunov, liên tục, afin từng mảnh. Từ đó, việc ước lượng được tiến hành một cách hiệu quả.
Từ khóa: *miền hấp dẫn, hệ phi ô-tô-nôm, bài toán quy hoạch tuyến tính, lý thuyết Lyapunov, hàm Lyapunov liên tục, afin từng mảnh.*

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