NEW RESULT ON INPUT-OUTPUT FINITE-TIME STABILITY OF FRACTIONAL-ORDER NEURAL NETWORKS

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SUMMARY

In this paper, we investigate the problem of input-output finite-time (IO-FT) stability for a class of fractional-order neural networks with a fractional commensurate order $0 < \alpha < 1$. By constructing a simple Lyapunov function and employing a recent result on Caputo fractional derivative of a quadratic function, new sufficient condition is established to guarantee the IO-FT stability of the considered systems. A numerical example is provided to illustrate the effectiveness of the proposed result.

Key words: *Fractional-order neutral networks; Input-output finite-time stability;Linear matrix inequality; Caputo derivative; Symmetric positive definite matrix.*

INTRODUCTION

Fractional-order neural networks have recently attracted an increasing attention in interdisciplinary areas by their wide applications to physics, biological neurons and intellectual intelligence. In the form of fractional-order derivative or integral, the neural networks are importantly improved in terms of the infinite memory and the hereditary properties of network processes. Besides, fractional-order differentiation is proved to provide neurons with the fundamental and general computation ability, facilitating the efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing. As a result, many interesting and important results on fractional-order neural networks have been obtained (see, [1], [2], [3] and references therein).

In many practical applications, it is desirable that the dynamical system possesses the property that its states do not exceed a certain threshold during a finite-time interval when given a bound on the initial condition. In these cases, finite-time stability concept could be used [4], [5]. Roughly speaking, fractional-order neural networks are said to be FT stable if the states do not beat some bounds within an arranged fixed time interval when the initial states satisfy a specified bound. It is important to recall that FT stability and Lyapunov asymptotic stability (LAS) are independent concepts; indeed a system can be FT stable but not LAS, and vice versa [6]. LAS concept requires that the systems operate over an infinite-time interval; meanwhile, all real neural systems operate over infinite-time interval. Therefore, it is necessary to care more about the finite-time behavior of systems than the asymptotic behavior over an infinite time interval. Some interesting results have been developed to treat the problem of finite-time stability of fractional-order neural networks systems in the literature [7], [8], [9]. By using the theory of fractional-order differential equations with discontinuous right-hand sides, Laplace transforms,Mittag-Leffler functions and generalized Gronwall inequality, the authors in [7] derived some sufficient conditions to guarantee the infinite-time stability of the fractional-order complex-valued memristor-based neural networks with time delays. Some delay-independent finite-time stability criteria were derived for fractional-order neural networks with delay in [8]. Recently, the problem of FT stability analysis for fractional-order Cohen-Grossberg BAM neural networks with time delays was considered in [9] by using some inequality

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techniques, differential mean value theorem and contraction mapping principle.

So far, the FT stability mainly concerns the specified bounds on the system states with a given initial bound; however, sometimes just the outputs, rather than the states, are required to be restrained within a bound. In this case, the IO-FT stability of a system is of significance. With regard to integer-order systems, the concept of IO-FT stability was originally introduced by Amato et.al in [10]. A system is IO-FT stability if, for a given class of input signals, the output of the system does not exceed an assigned threshold during a specified time interval. Up to now, some efforts have been devoted to the research of IO-FT stability for integer-order systems (see, [11], [12]). Regarding to fractional-order systems, to the best of our knowledge, there is only one result concerning the IO-FT stability of linear systems [13]. While FT stability analysis of fractional-order neural networks systems have been widely studied and developed, (see, [7], [8], [9] and the references therein), the problem of IO-FT stability of fractional-order neural networks has not been considered in the literature. This problem is challenging due to the complexity of fractional-order calculus equation and the fact that integer-order algorithms cannot be directly applied to the fractional-order systems. The aforementioned discussion inspires us for the present study.

In this paper, we study the problem of IO-FT stability of fractional-order neural networks. The main contributions of this work can be summarized as follow. By constructing a simple Lyapunov function and employing a recent result on Caputo fractional derivative of a quadratic function, we derive new sufficient condition guaranteeing the IO-FT stability of the considered systems. The condition is with the form of linear matrix inequalities (LMIs), which therefore can be effectively solved by using existing convex algorithms. Moreover, a numerical example is provided to show the effectiveness and applicability of the proposed scheme.

The remaining of this paper is organized as follows. Some necessary definitions and lemmas are recalled in the next section. Sufficient condition ensuring the IO-FT stability of fractional-order neural networks is shown in the Section 3. Finally, a numerical example is given to present the effectiveness of the scheme in the Section 4.

Notations**:** The following notations will be used in this paper: $\mathbb{R}^n$ denotes the $n-$dimensional linear vector space over the reals with the Euclidean norm (two-norm) $\|.\|$ given by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}, x = (x_1, \dots, x_n) \in \mathbb{R}^n$. $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices. For a real matrix A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and the minimal eigenvalue of A, resppectively. Matrix P is positive definite $(P > 0)$ if $x^T P x > 0, \forall x \neq 0$. $P > Q$ means $P - Q > 0$. The symbol $L_\infty \coloneqq L_\infty(T_f, R)$, where R is given symmetric positive definite matrix, refers to the space of essentially bounded signals, $\omega(.) \in L_\infty(T_f, R)$ if $\omega(t) \leq 1, \forall t \in [0, T_f]$.

PROBLEM STATEMENT AND PRELIMINARIES

To begin with, we recall the fundamental definition of the fractional calculus found in [14]. The fractional integral with non-integer order $\alpha > 0$ of a function $x(t)$ is defined as follows:

$$_{t_o}I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_o}^{t} (t-s)^{\alpha-1} x(s) ds,$$

where $x(t)$ is an arbitrary integrable function, $_{t_o}I_t^\alpha$ denotes the fraction integral of order α on $[t_o, t]$ and $\Gamma(.)$ represent the gamma function. The Caputo fractional-order derivative of order $\alpha > 0$ for a function $x(t) \in C^{n+1}([to, +\infty), \mathbb{R})$ is defined as follows:

$$ {}_{t_o}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_o}^{t}\frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, \qquad t \geq t_o \geq 0, $$

where n is a positive integer such that $n-1<\alpha<n$. In particular, when $0<\alpha<1$, we have:

$$ {}_{t_o}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_o}^{t}\frac{\dot{x}(s)}{(t-s)^{\alpha}}ds, \qquad t \geq t_o \geq 0. $$

Especially, as in [14], we have ${}_{t_o}^{C}D_t^{0}x(t) = x(t)$ and ${}_{t_o}^{C}D_t^{1}x(t) = \dot{x}(t)$. Let us now consider the following controlled Caputo fractional-order neural networks:

$$ \begin{cases} {}_{0}^{C}D_t^{\alpha}x(t) = -Ax(t) + Df\big(x(t)\big) + W\omega(t), t \geq 0 \\ y(t) = Cx(t), \\ x(0) = 0 \end{cases} \tag{1} $$

where $0<\alpha<1$ is the fractional commensurate order of the system, $x(t) = \big(x_1(t),\dots,x_n(t)\big)^T \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^q$ is the output vector, $\omega(t) \in \mathbb{R}^p$ is the disturbance input vector, n is the number of neural, $f(x(t)) = (f_1\big(x_1(t)\big),\dots,f_n\big(x_n(t)\big)))^T \in \mathbb{R}^n$ denotes the activation function, $A = diag\{a_1,\dots,a_n\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $D \in \mathbb{R}^{n\times n}$ is interconnection weight matrix, $C \in \mathbb{R}^{q\times n}, W \in \mathbb{R}^{n\times p}$ are known real matrices.

Assume that the activation function $f_i(.)$ is continuous, $f_i(0) = 0, (i = 1,\dots,n)$ and satisfies the following growth conditions with the growth known positive constants $\gamma_i(i = 1,\dots,n)$:

$$ |f_i(x) - f_i(y)| \leq \gamma_i|x-y|, (i = 1,\dots,n), \forall x,y \in \mathbb{R}. \tag{2} $$

In the case where $y = 0$, the condition (2) becomes:

$$ |f_i(x)| \leq \gamma_i|x|, (i = 1,\dots,n), \forall x \in \mathbb{R}. \tag{3} $$

Now, let us recall the following definition and some auxiliary lemmas which are essential in order to derive our main results in this paper.

Definition 1.([10]) Given a positive scalar $T_f > 0$, a symmetric positive definite matrix $Q \in \mathbb{R}^{q\times q}$, the system (1) is said to be IO-FT stable with respect to (L_∞, Q, T_f) if $\omega(.) \in L_\infty$, implies $y^T(t)Qy(t) < 1, \forall t \in [0, T_f]$.

Lemma 1.([14]) *If* $x(t) \in C^n([0,+\infty),\mathbb{R})$ *and* $n-1<\alpha<n, (n \geq 1, n \in \mathbb{Z}^+)$, *then*

$$ {}_{0}I_t^{\alpha}\left({}_{0}^{C}D_t^{\alpha}x(t)\right) = x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0). $$

In particular, when $0<\alpha<1$, *we have*

$$ {}_{0}I_t^{\alpha}\left({}_{0}^{C}D_t^{\alpha}x(t)\right) = x(t) - x(0). $$

Lemma 2.([15]). *Let* $x(t) \in \mathbb{R}^n$ *be a vector of differentiable function. Then, for any time instant* $t \geq t_0$, *the following relationship holds:*

$$ {}_{t_0}^{C}D_t^{\alpha}(x^T(t)Px(t)) \leq 2x^T(t)P\,{}_{t_0}^{C}D_t^{\alpha}x(t), \forall \alpha \in (0,1), \forall t \geq t_0 \geq 0. $$

MAIN RESULTS

The following theorem provides sufficient conditions under which the fractional-order neural networks (1) is IO-FT stability with respect to (L_∞, Q, T_f).

Theorem 1. *The fractional order neural networks (1) IO-FT stable with respect to* (L_∞, Q, T_f) *if there exist a symmetric positive definite matrix P and a diagonal positive matrix Λ such that the following conditions hold:*

$$\begin{bmatrix} M & PD & PW \\ * & -\Lambda & 0 \\ * & * & -R \end{bmatrix} < 0, \tag{4a}$$

$$C^T Q C < \frac{\Gamma(\alpha+1)}{T_f^\alpha} P, \tag{4b}$$

where $M = -PA - A^T P + H\Lambda H,\ H = diag\{\gamma_1, \dots, \gamma_n\}$.

Proof. We consider the following non-negative quadratic function:

$$V(x(t)) = x^T(t) P x(t).$$

It follows from Lemma 2 that the α −order ($0 < \alpha < 1$) Caputo derivative of $V(x(t))$ along the trajectories of system (1) is obtained as follows:

$$\begin{aligned} {}_0^C D_t^\alpha V x(t)) &\le 2x^T(t) P {}_0^C D_t^\alpha x(t) \\ &= x^T(t)[-PA - A^T P] x(t) + 2x^T(t) PDf(x(t)) + 2x^T PW\omega(t). \end{aligned} \tag{5}$$

The following inequalities are resulted from the Cauchy matrix inequality:

$$2x^T(t) PDf(x(t)) \le x^T(t) PD\Lambda^{-1} D^T P x(t) + f^T(x(t)) \Lambda f(x(t)) \tag{6a}$$

$$2x^T(t) PW\omega(t) \le x^T(t) PWR^{-1} W^T P x(t) + \omega^T(t) R \omega(t) \tag{6b}$$

Since Λ is a diagonal positive matrix, from (3), we have the following estimate:

$$f^T(x(t)) \Lambda f(x(t)) \le x^T(t) H\Lambda H x(t) \tag{7}$$

From (5) to (7), we obtain:

$${}_0^C D_t^\alpha V(x(t)) \le x^T(t) \Omega x(t) + \omega^T(t) R \omega(t), \tag{8}$$

where

$$\Omega = -PA - A^T P + PD\Lambda^{-1} D^T P + PWR^{-1} W^T P + H\Lambda H.$$

From Schur Complement Lemma, $\Omega < 0$ is equivalent to condition (4a), implying:

$${}_0^C D_t^\alpha V(x(t)) < \omega^T(t) R \omega(t), \forall t \in [0, T_f]. \tag{9}$$

Since $V(x(0)) = 0$, by integrating both sides of (9) (with order α) from 0 to t, ($0 < t < T_f$), and using Lemma 1, the following inequality is obtained:

$$\begin{aligned} x^T(t) P x(t) < {}_0 I_t^\alpha (\omega^T(t) R \omega(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega^T(s) R \omega(s) ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \le \frac{1}{\Gamma(\alpha+1)} T_f^\alpha. \end{aligned} \tag{10}$$

From (4b) and (10), we have:

$$y^T(t) Q y(t) = x^T(t) C^T Q C x(t) < \frac{\Gamma(\alpha+1)}{T_f^\alpha} x^T(t) P x(t) \le 1, \forall t \in [0, T_f],$$

which completes the proof of Theorem 1.

NUMERICAL EXAMPLES

The example below is presented to illustrate the effectiveness of the proposed method.

Example 1. Consider the following fractional-order neural networks:

$$\begin{cases} {}_0^C D_t^{\alpha} = -Ax(t) + Df(x(t)) + W\omega(t), t \geq 0 \\ y(t) = Cx(t) \\ x(0) = 0 \end{cases} \tag{11}$$

where $\alpha = 0.9; x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3, \omega(t) = e^{-t} \in \mathbb{R}$, the activation function are given by:

$$f_i(x_i(t)) = \frac{1}{2}(|x_i(t) + 1| - |x_i(t) - 1|, i = 1,2,3,$$

and

$$\boldsymbol{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 1.0 & 0.2 & 0.9 \\ 0.4 & 0.3 & 1.0 \\ 0.2 & 0.1 & 0.8 \end{bmatrix}, \boldsymbol{W} = \begin{bmatrix} 1.0 \\ 0.5 \\ 0.9 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.2 \end{bmatrix}^T.$$

It is easy to verify that condition (2) is satisfied with $H = diag\{1,1,1\}$. Given $T_f = 10, R = [1], Q = [1]$. By using Theorem 1, wefound that the LMI conditions of (4a) and (4b) are satisfied with

$$\boldsymbol{P} = \begin{bmatrix} 1.4053 & -0.0194 & -0.4344 \\ -0.0194 & 2.7889 & -0.7632 \\ -0.4344 & -0.7632 & 1.9241 \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} 2.0518 & 0 & 0 \\ 0 & 1.4932 & 0 \\ 0 & 0 & 2.1278 \end{bmatrix}.$$

Thus, system (11) is IO-FT stable with respect to (L_∞, Q, T_f) based on Theorem 1.

CONCLUSION

This paper has investigated the problem of IO-FT stability of fractional-order neural networks. Based on constructing a simple Lyapunov function and using some properties of Caputo fractional derivative, sufficient condition for the IO-FT stability of the considered systems is derived in the form of linear matrix inequalities, which therefore can be effectively solved by using existing convex algorithms. The effectiveness of the result has been demonstrated via the numerical example.

REFERENCES

1. H. Wang, Y. Yu, G. Wen, S. Zhang (2015), "Stabilitity analysis of fractional-order neural networks with time delays", *Neural Process Lett.*, 42, pp. 479-500.
2. H. Wang, Y. Yu, G. Wen, S. Zhang, J. Yu (2015), "Global stability analysis of fractional-order Hopfield neural networks with time delays", *Neurocomputing*, 154, pp. 15-23.
3. S. Zhang, Y. Yu, H. Wang (2015), "Mittag-Leffler stability of fractional-order Hopfield neural networks", *Nonlinear Analysis: Hybrid Syst.*, 16, pp. 104-121.
4. P. Dorato (1961), "Short time stability in linear time-varying system", *Proceeding of the IRE International Convention Record Part 4, New York*, pp. 83-87.
5. L. Weiss, E.F. Infante (1967), "Finite time stability under perturbing forces and product spaces", *IEEE Tran. on Automatic Cont.*, 12, pp. 54-59.
6. F. Amato, M. Ariola, C. Cosentino (2006), "Finite-time stabilization via dynamic output feedback", *Automatica*, 42, pp. 337-342.
7. R. Rakkiyappan, G. Velmurugan, R. Cao (2014), "Finite-time stability criteria for a class of fractional-order complex-valued memristor-based neural networks with time delays", *Nonlinear Dynamics*, 78, pp. 2823-2836.
8. X. Yang, Q. Song, Y. Liu, Z. Zhao (2015), "Finite-time stability analysis of fractional-order neural networks with delay", *Neurocomputing*, 152, pp. 19-26.
9. C. Rajivganthi, F.A. Rihan, S. Lakshmanan, P. Muthukumar (2016), "Finite-time stability analysis for fractional-order Cohen–Grossberg BAM neural networks with time delays", *Neural computing and Appl.*, 29, pp. 1309-1320.
10. F. Amato, R. Ambrosino, C. Cosentino, G. De Tommasi (2010), "Input-output finite time stabilization of linear systems", *Automatica*, 46, pp. 1558-1562.

11. F. Amato, G. carannante, G. De Tommasi, A. Pironti (2012), "Input-output finite time stability of linear systems:Necessary and sufficient conditions", *IEEE Tran. on Automatic Cont.*, 57, pp. 3051-3063.
12. F. Amato, G. De Tommasi, A. Pironti (2016), "Input-output finite-time stabilisation of impulsive linear systems: Necessary and sufficient conditions", *Nonlinear Analysis: Hybrid Syst.*, 19, pp. 93-106.
13. Y. Ma, B. Wu, Y.E. Wang, Y. Cao (2017), "Input-output finite-time stability of fractional order linear systems with $0 < \alpha < 1$", *Transactions of the Institute of Measurement and Control*, 39, pp. 653-659.
14. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo (2006), *Theory and applications of fractional differential equations*, Elsevier, Amsterdam.
15. M.A. Duarte-Mermoud, N. Aguila-Camacho, J.A. Gallegos, R. Castro-Linares (2015), "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems", *Communications in Nonlinear Science and Numerical Simulation*, 22, pp. 650-659.

TÓM TẮT

KẾT QUẢ MỚI VỀ TÍNH ỔN ĐỊNH HỮU HẠN THỜI GIAN ĐẦU VÀO-ĐẦU RA CỦA HỆ NƠ RON THẦN KINH PHÂN THỨ

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Trong bài báo này, chúng tôi nghiên cứu bài toán ổn định hữu hạn thời gian đầu vào-đầu ra cho một lớp hệ nơron thần kinh phân thứ. Bằng cách xây dựng một hàm Lyapunov đơn giản và sử dụng một kết quả gần đây về tính đạo hàm phân thứ Caputo của một hàm toàn phương, chúng tôi đưa ra một điều kiện đủ cho tính ổn định hữu hạn thời gian đầu vào-đầu ra của lớp hệ nơron thần kinh phân thứ. Một ví dụ số được đưa ra để minh họa tính hiệu quả của kết quả do chúng tôi đề xuất.

Từ khóa: *Hệ nơron thần kinh phân thứ; Ổn định hữu hạn thời gian đầu vào-đầu ra; Bất đẳng thức ma trận tuyến tính; Đạo hàm Caputo; Ma trận đối xứng xác định dương.*

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