THE CHARACTER FORMULA OF THE LIE SUPER AlGEBRA $gl(2|2|)$

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Tóm tắt

Công thức đặc trưng của các biểu diễn bất khả quy điển hình đã được Kac đưa ra trong [6]. Cho đến 2010, trong [10] Zhang đã đưa ra được công thức tổng quát cho các biểu diễn không điển hình. Tuy vậy, các công thức này cực kỳ là phức tạp, để hiểu được bản chất và tính được tường minh các đặc trưng của biểu diễn là công việc rất khó. Nội dung chính trong bài báo này, chúng tôi tìm cách tính toán một cách tường minh đặc trưng của tất cả các biểu diễn bất khả qui của siêu đại số Lie $gl(2|2|)$.

Từ khóa: *Đa thức đặc trưng, công thức đặc trưng, biểu diễn bất khả quy.*

1 Introduction

The formula character of finite dimensional irreducible representations of complex simple Lie superalgebra encapsulate rich information on the structure of the representations themselves. Kac raised the problem determining the character formula for the so - called typical irreducible representations. However, the problem turned out to be quite hard for the so-called atypical irreducible representation. In the early 80s Bernstein and Leites gave a formula for the general linear superalgebra, which produces the correct formula characters for the singly atypical irreducible representations, but fail for the multiply atypical irreducible. since then much further research was done on the problem. Particularly important is the work of Brudan, who developed a very practicable algorithm for computing the generalized Kazhdan-Lusztig polynomial by using quantum group techniques.

In [9] Zhang shall further investigate Brudan's algorithm and implement it to compute the generalized Kazhdan-Lusztig polynomials for the finite dimensional irreducible representations of the general linear superalgebra. A closed formulla is obtained for the generalized Kazhdan-Lusztig polynomials, which is essentially given in terms of the permutations group of the atypical roots.

In [10] Zhang gave out the character and dimension formula of typical and atypical irreducible representations for the general linear superalgebras. Therefore, in the case of $gl(2|2)$, by using the result of Zhang and Kac, we will propose the explicit formula of all finite irreducible representations.

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2 Lie superalgebra

Let V be a super vector space with super dimension $(m|n)$. $V = V_0 + V_1$, we call V_0, V_1 the even and odd subspace, respectively. Define a map $[]: V_{\bar{0}} \bigoplus V_{\bar{1}} \to Z_2$ by $[\bar{v}] = i$ if $v \in V_{\bar{i}}$. For any two Z_2-graded vector space V and W the space of morphism $Hom_C(V, W) = \sum_{i+j=k} Hom(V_i, W_j)$. We write $End_C(V)$ for $Hom(V, V)$.

Let $C^{m|n|}$ be the Z-graded vector space with even subspace C^m and odd subspace C^n. Then $End(C^{m|n|})$ with the Z_2-graded commutator form the general linear superalgebra. To describe its structure, we choose a homogeneous basis $\{v_a | a \in I\}$ for $C^{m|n|}$, where $I = \{1, 2, \cdots m+n\}$ and v_a is even if $a \leq m$, and old otherwise. The general linear superalgebra relative to this basis of $C^{m|n|}$ will be denoted $gl_{m|n|}$, which shall be further simplified to g through the paper. Let E_{ab} be the matrix unit, namely, the $m+n, m+n$-matrix with all entries being zero except that at the (a, b) position is 1. Then $\{E_{ab} | a, b \in I\}$ forms a homogenerous basis of g, with E_{ab} being even if $a, b \leq m$ or $a, b > m$, and odd otherwise. For convenience, we define the map $[]: I \longrightarrow Z_2$ by $[a] = \bar{0}$ if $a \leq m$ and $[a] = 1$ if $a > m$.

Then the commutation relations of the Lie Superalgebras can be written associated

$$[E_{ab}, E_{cd}] = E_{ad}\delta bc - (-1)^{([a]-[b])([c]-[d])} E_{cb}\delta_{ad}.$$

The upper triangular matrices form a *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$, which contains the *Cartan subalgebra* $\mathfrak{h}$ of diagonal matrices. Let $\{\epsilon_a | a \in I\}$ be the basis of $\mathfrak{h}^*$ such that $\epsilon_a(E_{ab}) = \delta_{ab}$. The supertrace induces a bilinear form $(,): \mathfrak{h}^* x \mathfrak{h}^* \longrightarrow C$ on $\mathfrak{h}^*$ such that

$$(\epsilon_a, \epsilon_b) = (-1)^{[a]}\delta_{ab}.$$

Relative to the Borel subalgebra $\mathfrak{b}$, the roots of $\mathfrak{g}$ can be expressed as

$\epsilon_a - \epsilon_b, a \neq b$, where $\epsilon_a - \epsilon_b$ is even if $[a] + [b] = 0$

and odd otherwise. The set of the *positive roots* is $\Delta^+ = \{\epsilon_a - \epsilon_b | a < b\}$ and the set of *simple roots* is $\{\epsilon_a - \epsilon_{a+1} | a < m+n\}$.

We denote $I^1 = \{1, 2, \cdots, m\}$ and $I^2 = \{1.2. \cdots, n\}$. We also set $\delta_\xi = \epsilon_{\xi}$ for $\xi \in I^2$, where we use the notation $\xi = \xi + m$.

Then the set of *positive even roots* and *odd* are respectively

$$\Delta_0^+ = \{\epsilon_i - \epsilon_j, \delta_\xi - \delta_\eta | 1 \neq i \neq j \neq m, 1 \neq \xi < \eta \neq n\},$$

$$\Delta_1^+ = \{\epsilon_i - \delta_\xi | i \in I^1, \xi \in I^2\}.$$

An element in $\mathfrak{h}^*$ is called a *weight*. A weight $\lambda \in \mathfrak{h}^*$ will be written in terms of the $\epsilon\delta$-basis as

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m | \lambda_{\bar{1}}, \lambda_{\bar{2}}, \cdots, \lambda_{\bar{n}}) = \sum_{i \in I^1} \lambda_i \epsilon_i - \sum_{\xi \in I^2} \lambda_\xi \delta_\xi$$

3 The characters of irreducible representations of $gl(2|2|)$

Representation of simple Lie superalgebras have been classified by Kac In [6], Kac proved that any finite dimensional irreducible representations of Lie super algebra $\mathfrak{gl}(m|n)$ is one of Verma module V_λ. He divided irreducible representations of $\mathfrak{gl}(m|n)$ into two classes, typical representations and atypical representations. By using Verma module, Kac is able to give explicit construction of all typical representation of $\mathfrak{gl}(m|n)$. A character formula and an explicit construction for all typical representation was also obtained. But a character formula and explicit construction of all atypical representation have not be obtained. In [10] R.B.Zhang gave a character formula for all finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$.

It is known that irreducible representations the super group $GL_C(m|n)$ are in 1-1 corresponding with irreducible $\mathfrak{gl}(m|n)$-modules with integrable highest weight [9]. The main purpose of this work is to explicitly compute the character formula of all irreducible representations of $gl(2|2)$ In [10], we have the following Theorem 4.9. In $gl(2|2)$, we have

$$\Delta_1^+ = \{\epsilon_1 - \delta_1, \epsilon_1 - \delta_2, \epsilon_2 - \delta_1, \epsilon_2 - \delta_2\}$$

$$\Delta_0^+ = \{\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}; \cdots \rho = (1/2, -1/2, -3/2).$$

We denote $R := (x_1 + y_1)(x_1 + y_2)(x_2 + y_1)(x_2 + y_2), \Pi = (x_1 - x_2)(y_1 - y_2)$.

According to [10], we have the following notations:

For $s <= t$, we say that atypical roots γ_s, γ_t are $c-$related if $s = t$ or $\lambda_{m_t} - \lambda_{m_s} < n_t.n_s$.

$c_{s,t} = 1$ if the atypical γ_s, γ_t are $c-$related; $c_{s,t} = 0$ otherwise.

Weigh λ is called totally disconnected if $c_{s,t}(\lambda) = 0$ for all $(s,t) : s < t$.

Weigh λ is called totally connected if $c_{s,t}(\lambda) = 1$ for all $(s,t) : s <= t$.

According to [5], we have: Let λ be a atypical weigh, then $\lambda = (\lambda_1, \lambda_2|\lambda_3, \lambda_4)$, where λ satisfies one of the following cases

- $\lambda_1 = \lambda_4$ and $\lambda_2 \neq \lambda_4$: that means $\epsilon_1 - \delta_2$ is a atypical root;
- $\lambda_2 = \lambda_3$ and $\lambda_1 \neq \lambda_4$: that means $\epsilon_2 - \delta_1$ is a atypical root;
- $\lambda_1 + 1 = \lambda_3$: that means $\epsilon_1 - \delta_1$ is a atypical root;
- $\lambda_4 + 1 = \lambda_2$: that means $\epsilon_2 - \delta_2$ is a atypical root;
- $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$: that means $\epsilon_1 - \delta_2, \epsilon_2 - \delta_1$ are atypical roots.

We denote $x_1 := e^{\epsilon_1}, x_2 := e^{\epsilon_2}$ $\quad y_1 := e^{\delta_1}, y_2 := e^{\delta_2}$.

Then, According to Theorem 4.9 in [6] and [10], we will proved the following theorem.

Theorem 3.1. *Let $\lambda = (\lambda_1, \lambda_2|\lambda_3, \lambda_4)$ be a weigh, then we have*

- *If λ is a typical weigh,*

$$chV(\lambda) = \frac{R}{\Pi(x_1x_2y_1y_2)^{\frac{1}{2}}}\Big[\frac{x_1^{\lambda_1-1/2}.x_2^{\lambda_2-3/2}}{y_1^{\lambda_3-3/2}y_2^{\lambda_4-1/2}} - \frac{x_1^{\lambda_1-1/2}.x_2^{\lambda_2-3/2}}{y_1^{\lambda_4-1/2}y_2^{\lambda_3-3/2}} + \frac{x_1^{\lambda_2-3/2}.x_2^{\lambda_1-1/2}}{(y_1^{\lambda_4-1/2}y_2^{\lambda_2-3/2}} - \frac{x_1^{\lambda_2-3/2}.x_2^{\lambda_1-1/2}}{y_1^{\lambda_3-3/2}y_2^{\lambda_4-1/2}}\Big];$$

- *If λ be a atypical weigh, we have the following cases:*
  - *If $\lambda_1 = \lambda_4, \lambda_2 \neq \lambda_3$, we have*

$$chV(\lambda) = \frac{R}{\Pi}\Big[\frac{x_1^{\lambda_1}.x_2^{\lambda_2-2}}{(x_1+y_2)y_1^{\lambda_3-1}y_2^{\lambda_1}} - \frac{x_1^{\lambda_1}.x_2^{\lambda_2-2}}{(x_1+y_1)y_1^{\lambda_1}y_2^{\lambda_3-1}} + \frac{x_1^{\lambda_2-2}.x_2^{\lambda_1}}{(x_2+y_1)y_1^{\lambda_1}y_2^{\lambda_3-1}} - \frac{x_1^{\lambda_2-2}.x_2^{\lambda_1}}{(x_2+y_2)y_1^{\lambda_3-1}y_2^{\lambda_1}}\Big];$$

  - *If $\lambda_2 = \lambda_3, \lambda_1 \neq \lambda_4$, we have*

$$chV(\lambda) = \frac{R}{\Pi}\Big[\frac{x_1^{\lambda_1-1}.x_2^{\lambda_2-1}}{(x_2+y_1)y_1^{\lambda_2-1}y_2^{\lambda_4}} - \frac{x_1^{\lambda_1-1}.x_2^{\lambda_2-1}}{(x_2+y_2)y_1^{\lambda_4}y_2^{\lambda_2-1}} + \frac{x_1^{\lambda_2-1}.x_2^{\lambda_1-1}}{(x_1+y_2)y_1^{\lambda_4}y_2^{\lambda_2-1}} - \frac{x_1^{\lambda_2-1}.x_2^{\lambda_1-1}}{(x_1+y_1)y_1^{\lambda_2-1}y_2^{\lambda_4}}\Big];$$

  - *If $\lambda_1 + 1 = \lambda_3$, we have*

$$chV(\lambda) = \frac{R}{\Pi}\Big[\frac{x_1^{\lambda_1}.x_2^{\lambda_2-2}}{(x_1+y_1)y_1^{\lambda_1}y_2^{\lambda_4}} - \frac{x_1^{\lambda_1}.x_2^{\lambda_2-2}}{(x_1+y_2)y_1^{\lambda_4}y_2^{\lambda_1}} + \frac{x_1^{\lambda_2-2}.x_2^{\lambda_1}}{(x_2+y_2)y_1^{\lambda_4}y_2^{\lambda_1}} - \frac{x_1^{\lambda_2-2}.x_2^{\lambda_1}}{(x_2+y_1)y_1^{\lambda_1}y_2^{\lambda_4}}\Big];$$

  - *If $\lambda_2 = \lambda_4 + 1$, we have*

$$chV(\lambda) = \frac{R}{\Pi}\Big[\frac{x_1^{\lambda_1-1}.x_2^{\lambda_4}}{(x_2+y_2)y_1^{\lambda_3-1}y_2^{\lambda_4}} - \frac{x_1^{\lambda_1-1}.x_2^{\lambda_4}}{(x_2+y_1)y_1^{\lambda_4}y_2^{\lambda_3-1}} + \frac{x_1^{\lambda_4}.x_2^{\lambda_1-1}}{(x_1+y_1)y_1^{\lambda_4}y_2^{\lambda_3-1}} - \frac{x_1^{\lambda_4}.x_2^{\lambda_1-1}}{(x_1+y_2)y_1^{\lambda_3-1}y_2^{\lambda_4}}\Big];$$

  - *If $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$, we have*

$$chV(\lambda) = \frac{R}{\Pi}\Big[\frac{x_1^{\lambda_1}.x_2^{\lambda_2-1}}{(x_2+y_1)(x_1+y_2)y_1^{\lambda_2-1}y_2^{\lambda_1}} - \frac{x_1^{\lambda_1}.x_2^{\lambda_2-1}}{(x_2+y_2)(x_1+y_1)y_1^{\lambda_1}y_2^{\lambda_2-1}} + \frac{x_1^{\lambda_2-1}.x_2^{\lambda_1}}{(x_1+y_2)(x_2+y_1)y_1^{\lambda_1}y_2^{\lambda_2-1}} - \frac{x_1^{\lambda_2-1}.x_2^{\lambda_1}}{(x_1+y_1)(x_2+y_2)y_1^{\lambda_2-1}y_2^{\lambda_1}}\Big];$$

Tài liệu

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SUMMARY: In this paper, we explicitely compute the character formula for all irreducible representations of Lie super algebra $gl(2|2)$.

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