WEAK AND STRONG CONVERGENCE FOR NONEXPANSIVE NONSELF-MAPPING

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Abstract: Suppose C is a nonempty closed convex nonexpansive retract of real uniformly convex Banach space X with P a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping of C with $F(T) := \{x \in C : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

$$y_n = P((1 - a_n - \mu_n)x_n + a_n TP((1 - \beta_n)x_n + \beta_n Tx_n) + \mu_n w_n),$$
$$x_{n+1} = P((1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n),\ n \geq 1,$$

where $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and$\{\gamma_n\}$ are appropriate sequences in $[0, 1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C. (1) If T is a completely continuous nonexpansive nonself-mapping, then strong convergence of $\{x_n\}$ to some $x^* \in F(T)$ is obtained; (2) If T satisfies condition, then strong convergence of $\{x_n\}$ to some $x^* \in F(T)$ is obtained; (3) If X is a uniformly convex Banach space which satisfies Opial's condition, then weak convergence of $\{x_n\}$ to some $x^* \in F(T)$ is proved.

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1 Introduction

Fixed point iteration processes for approximating fixed points of nonexpansive mappings in Banach spaces have been studied by various authors (see [3, 4, 6, 9, 10, 15, 17, 19]) using the Mann iteration process (see [6]) or the Ishikawa iteration process (see [3, 4, 15, 19]). For nonexpansive nonself-mappings, some authors (see [19, 12, 14, 16]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces. In 2000, Noor [7] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 1998, Takahashi and Kim [14] proved strong convergence of approximants to fixed points of nonexpansive nonself-mappings in reflexive Banach spaces with a uniformly Gâteaux differentiable norm. In the same year, Jung and Kim [5] proved the existence of a fixed point for a nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. In [15], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space X. More preciesely, they proved the following theorem.

Theorem 1.1. [15]. *Let X be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm and C a nonempty closed convex bounded subset of X. Let $T : C \to C$ be a nonexpansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) < \infty$, and $\limsup_{n\to\infty} \beta_n < 1$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n T x_n), \quad n \geq 1 \tag{1.1}$$

converges weakly to some fixed point of T.

Suantai [13] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, Shahzad [12] extended Tan and Xu results [15] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space.

Inspired and motivated by research going on in this area, we define and study a new iterative scheme with errors for nonexpansive nonself-mapping. This scheme can be viewed as an extension for the iterative scheme of Shahzad [12]. The scheme is defined as follows:

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ the nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative scheme:

$$\begin{aligned} y_n &= P((1-a_n-\mu_n)x_n + a_n TP((1-\beta_n)x_n + \beta_n T x_n) + \mu_n w_n), \\ x_{n+1} &= P((1-b_n-\delta_n)x_n + b_n TP((1-\gamma_n)y_n + \gamma_n T y_n) + \delta_n v_n), \end{aligned} \tag{1.2}$$

$n \geq 1$, where $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0,1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C. If $a_n = \mu_n = \delta_n \equiv 0$, then (1.2) reduces to the iterative scheme defined by Shahzad [12]:

$$x_1 \in C, \ x_{n+1} = P((1-b_n)x_n + b_n TP((1-\gamma_n)x_n + \gamma_n T x_n)) \ \ \forall n \geq 1, \tag{1.3}$$

where $\{b_n\}$ and $\{\gamma_n\}$, are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$.

If $T : C \to C$ and $a_n = \mu_n = \delta_n \equiv 0$, then (1.2) reduces to the iterative scheme (1.1) defined by Tan and Xu [15].

The purpose of this paper is to construct an iteration scheme for approximating a fixed point of nonexpansive nonself-mappings (when such a fixed point exists) and to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space. Our results extend and improve the corresponding ones announced by Shahzad [12], Tan and Xu [15], and others.

Now, we recall the well known concepts and results.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P.

Recall that a Banach space X is said to satisfy *Opial's condition* [8] if $x_n \to x$ weak as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [11] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

for all $x \in C$; (see [11]) for an example of nonexpansive mappings satisfying condition (A).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.2. [15] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \ \ \forall n = 1, 2, ...,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ exists.
(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.3. [17] *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda).$$

Lemma 1.4. [2] *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, and all $\alpha, \beta, \gamma, \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 1.5. [1] *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weak and $(x_n - Tx_n) \to 0$ strong, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 1.6. [13] *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2 Main Results

In this section, we prove weak and strong convergence theorems of the new iterative scheme (1.2) for nonexpansive nonself-mapping in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}$, $\{\mu_n\}$, $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.2). Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*

Proof. Let $x^* \in F(T)$, and

$$M = \max\{\sup_{n\geq 1} \|w_n - x^*\|, \ \sup_{n\geq 1} \|v_n - x^*\|\}.$$

For each $n \geq 1$, using (1.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P((1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n) - x^*\| \\
&= \|P((1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n) - P(x^*)\| \\
&\leq \|(1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n - x^*\| \\
&= \|(1 - b_n - \delta_n)(x_n - x^*) + b_n(TP((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*) + \delta_n(v_n - x^*)\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|TP((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|P((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|(1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n - x^*\| + \delta_n\|v_n - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&= (1-b_n-\delta_n)\|x_n-x^*\|+b_n\|(1-\gamma_n)(y_n-x^*) \\
&\quad +\gamma_n(Ty_n-x^*)\|+\delta_n\|v_n-x^*\| \\
&\le (1-b_n-\delta_n)\|x_n-x^*\|+b_n((1-\gamma_n)\|y_n-x^*\| \\
&\quad +\gamma_n\|y_n-x^*\|)+\delta_n\|v_n-x^*\| \\
&= (1-b_n-\delta_n)\|x_n-x^*\|+b_n\|y_n-x^*\|+\delta_n\|v_n-x^*\| \\
&\le (1-b_n-\delta_n)\|x_n-x^*\|+b_n\|y_n-x^*\|+M\delta_n \qquad (2.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n-x^*\| &= \|P((1-a_n-\mu_n)x_n+a_nTP((1-\beta_n)x_n+\beta_nTx_n)+\mu_nw_n)-x^*\| \\
&= \|P((1-a_n-\mu_n)x_n+a_nTP((1-\beta_n)x_n+\beta_nTx_n)+\mu_nw_n)-P(x^*)\| \\
&\le \|(1-a_n-\mu_n)x_n+a_nTP((1-\beta_n)x_n+\beta_nTx_n)+\mu_nw_n-x^*\| \\
&= \|(1-a_n-\mu_n)(x_n-x^*)+a_n(TP((1-\beta_n)x_n \\
&\quad +\beta_nTx_n)-x^*)+\mu_n(w_n-x^*)\| \\
&\le (1-a_n-\mu_n)\|x_n-x^*\|+a_n\|TP((1-\beta_n)x_n \\
&\quad +\beta_nTx_n)-x^*\|+\mu_n\|w_n-x^*\| \\
&\le (1-a_n-\mu_n)\|x_n-x^*\|+a_n\|P((1-\beta_n)x_n \\
&\quad +\beta_nTx_n)-x^*\|+\mu_n\|w_n-x^*\| \\
&\le (1-a_n-\mu_n)\|x_n-x^*\|+a_n\|(1-\beta_n)x_n \\
&\quad +\beta_nTx_n-x^*\|+\mu_n\|w_n-x^*\| \\
&= (1-a_n-\mu_n)\|x_n-x^*\|+a_n\|(1-\beta_n)(x_n-x^*) \\
&\quad +\beta_n(Tx_n-x^*)\|+\mu_n\|w_n-x^*\| \\
&\le (1-a_n-\mu_n)\|x_n-x^*\|+a_n(1-\beta_n)\|x_n-x^*\| \\
&\quad +a_n\beta_n\|x_n-x^*\|+\mu_n\|w_n-x^*\| \\
&= (1-a_n-\mu_n)\|x_n-x^*\|+a_n\|x_n-x^*\|+\mu_n\|w_n-x^*\| \\
&= (1-\mu_n)\|x_n-x^*\|+\mu_n\|w_n-x^*\| \\
&\le \|x_n-x^*\|+M\mu_n. \qquad (2.2)
\end{aligned}
$$

Using (2.1) and (2.2), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\le (1-b_n-\delta_n)\|x_n-x^*\|+b_n(\|x_n-x^*\|+M\mu_n)+M\delta_n \\
&= (1-b_n-\delta_n)\|x_n-x^*\|+b_n\|x_n-x^*\|+Mb_n\mu_n+M\delta_n \\
&= (1-\delta_n)\|x_n-x^*\|+Mb_n\mu_n+M\delta_n \\
&\le \|x_n-x^*\|+k_{(1)}^n, \qquad (2.3)
\end{aligned}
$$

where $k_{(1)}^n = Mb_n\mu_n + M\delta_n$.

Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, we have $\sum_{n=1}^{\infty}k_{(1)}^n < \infty$. We obtained from (2.3) and Lemma 1.2 that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. This completes the proof. □

Lemma 2.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ and $\{w_n\}, \{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} b_n$, and $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.2). Then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Let $x^* \in F(T)$. Then, by Lemma 2.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = r$. If $r = 0$, then by the continuity of T the conclusion follows. Now suppose $r > 0$. We claim

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Set $q_n = P((1-\beta_n)x_n + \beta_n Tx_n)$ and $s_n = P((1-\gamma_n)y_n + \gamma_n Ty_n)$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $x_n - x^*,\ y_n - x^* \in B_R(0)$ for all $n \geq 1$. Using Lemma 1.3, Lemma 1.4 and T is a nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P((1-b_n-\delta_n)x_n + b_n TP((1-\gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n) - x^*\|^2 \\
&= \|P((1-b_n-\delta_n)x_n + b_n Ts_n + \delta_n v_n) - x^*\|^2 \\
&\leq \|(1-b_n-\delta_n)x_n + b_n Ts_n + \delta_n v_n - x^*\|^2 \\
&= \|(1-b_n-\delta_n)(x_n - x^*) + b_n(Ts_n - x^*) + \delta_n(v_n - x^*)\|^2 \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + \delta_n\|v_n - x^*\|^2 \\
&\quad -(1-b_n-\delta_n)b_n g(\|Ts_n - x_n\|) \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + M^2\delta_n, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
\|Ts_n - x^*\|^2 &= \|TP((1-\gamma_n)y_n + \gamma_n Ty_n) - x^*\|^2 \\
&\leq \|P((1-\gamma_n)y_n + \gamma_n Ty_n) - x^*\|^2 \\
&\leq \|(1-\gamma_n)y_n + \gamma_n Ty_n - x^*\|^2 \\
&\leq \|(1-\gamma_n)(y_n - x^*) + \gamma_n(Ty_n - x^*)\|^2 \\
&\leq (1-\gamma_n)\|y_n - x^*\|^2 + \gamma_n\|Ty_n - x^*\|^2 \\
&\quad -W_2(\gamma_n)g(\|Ty_n - y_n\|) \\
&\leq \|y_n - x^*\|^2 - W_2(\gamma_n)g(\|Ty_n - y_n\|) \\
&\leq \|y_n - x^*\|^2, \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P((1-a_n-\mu_n)x_n + a_n TP((1-\beta_n)x_n + \beta_n Tx_n) + \mu_n w_n) - x^*\|^2 \\
&= \|P((1-a_n-\mu_n)x_n + a_n Tq_n + \mu_n w_n) - x^*\|^2 \\
&\leq \|(1-a_n-\mu_n)x_n + a_n Tq_n + \mu_n w_n - x^*\|^2 \\
&= \|(1-a_n-\mu_n)(x_n - x^*) + a_n(Tq_n - x^*) + \mu_n(w_n - x^*)\|^2 \\
&\leq (1-a_n-\mu_n)\|x_n - x^*\|^2 + a_n\|Tq_n - x^*\|^2 + \mu_n\|w_n - x^*\|^2 \\
&\quad -a_n(1-a_n-\mu_n)g(\|Tq_n - x_n\|) \\
&\leq (1-a_n-\mu_n)\|x_n - x^*\|^2 + \|Tq_n - x^*\|^2 + M^2\mu_n, \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
\|Tq_n - x^*\|^2 &= \|TP((1-\beta_n)x_n + \beta_n Tx_n) - x^*\|^2 \\
&\leq \|P((1-\beta_n)x_n + \beta_n Tx_n) - x^*\|^2 \\
&\leq \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\
&\quad -W_2(\beta_n)g(\|Tx_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 - W_2(\beta_n)g(\|Tx_n - x_n\|). \qquad (2.7)
\end{aligned}$$

By using (2.4), (2.5), (2.6) and (2.7), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + \delta_n M^2 \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|y_n - x^*\|^2 + \delta_n M^2 \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n((1-a_n-\mu_n)\|x_n - x^*\|^2 \\
&\quad +\|Tq_n - x^*\|^2 + M^2\mu_n) + M^2\delta_n \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n((1-a_n-\mu_n)\|x_n - x^*\|^2 \\
&\quad +(\|x_n - x^*\|^2 - W_2(\beta_n)g(\|Tx_n - x_n\|)) + M^2\mu_n) + M^2\delta_n \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n((1-a_n-\mu_n)\|x_n - x^*\|^2 \\
&\quad +\|x_n - x^*\|^2 - W_2(\beta_n)g(\|Tx_n - x_n\|) + M^2\mu_n) + M^2\delta_n \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n(\|x_n - x^*\|^2 \\
&\quad -W_2(\beta_n)g(\|Tx_n - x_n\|) + M^2\mu_n) + M^2\delta_n \\
&\leq \|x_n - x^*\|^2 - b_nW_2(\beta_n)g(\|Tx_n - x_n\|) \\
&\quad +M^2\mu_n + M^2\delta_n \\
&= \|x_n - x^*\|^2 - b_nW_2(\beta_n)g(\|Tx_n - x_n\|) + k^n_{(2)} \\
&= \|x_n - x^*\|^2 - b_n\beta_n(1-\beta_n)g(\|Tx_n - x_n\|) + k^n_{(2)}, \qquad (2.8)
\end{aligned}$$

where $k^n_{(2)} = M^2\mu_n + M^2\delta_n$. Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, we have $\sum_{n=1}^{\infty}k^n_{(2)} < \infty$. Since $0 < \liminf_{n\to\infty} b_n$ and $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$, there exists $n_0 \in \mathbb{N}$ and $\eta_1, \eta_2, \eta_3 \in (0,1)$ such that $0 < \eta_1 < b_n$ and $0 < \eta_2 < \beta_n < \eta_3 < 1$ for all $n \geq n_0$. It follows from (2.8) that

$$\eta_1\eta_2(1-\eta_3)g(\|Tx_n - x_n\|) \leq (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + k^n_{(2)},$$

for all $n \geq n_0$. Applying for $m \geq n_0$, we have

$$\begin{aligned}
\eta_1\eta_2(1-\eta_3)\sum_{n=n_0}^{m} g(\|Tx_n - x_n\|) &\leq \sum_{n=n_0}^{m}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \sum_{n=n_0}^{m} k^n_{(2)} \\
&= \|x_{n_0} - x^*\|^2 + \sum_{n=n_0}^{m} k^n_{(2)}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} k_{(2)}^{n} < \infty$, by letting $m \to \infty$ we get $\sum_{n=1}^{\infty} g(\|Tx_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|Tx_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. This completes the proof. □

Theorem 2.3. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}$, $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} b_n$, and $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.2) converge strongly to a fixed point of T.*

Proof. By Lemma 2.2, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| \;=\; 0. \tag{2.9}$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (2.9), $\{x_{n_k}\}$ converges. Let $q = \lim_{k\to\infty} x_{n_k}$. By the continuity of T and (2.9) we have that $Tq = q$, so q is a fixed point of T. By Lemma 1.2,$\lim_{n\to\infty} \|x_n - q\|$ exists. Then $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$. Using (1.2), we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1 - a_n - \mu_n)x_n + a_nTP((1 - \beta_n)x_n + \beta_nTx_n) + \mu_nw_n) - x_n\| \\
&\leq \|(1 - a_n - \mu_n)x_n + a_nTP((1 - \beta_n)x_n + \beta_nTx_n) + \mu_nw_n - x_n\| \\
&= \|a_n(TP((1 - \beta_n)x_n + \beta_nTx_n) - x_n) + \mu_n(w_n - x_n)\| \\
&= \|a_n(TP((1 - \beta_n)x_n + \beta_nTx_n) - Tx_n + Tx_n - x_n) + \mu_n(w_n - x_n)\| \\
&\leq a_n\|TP((1 - \beta_n)x_n + \beta_nTx_n) - Tx_n + Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\
&\leq a_n\|TP((1 - \beta_n)x_n + \beta_nTx_n) - Tx_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\
&\leq a_n\|P((1 - \beta_n)x_n + \beta_nTx_n) - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\
&\leq a_n\|(1 - \beta_n)x_n + \beta_nTx_n - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\
&\leq a_n\beta_n\|Tx_n - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that $\lim_{n\to\infty} \|y_n - q\| = 0$. This completes the proof. □

The following result gives a strong convergence theorem for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying *condition*(A).

Theorem 2.4. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}$, $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} b_n$, and $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. Suppose that T satisfies condition (A). Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.2) converge strongly to a fixed point of T.*

Proof. Let $x^* \in F(T)$. Then, as in Lemma 2.1, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and

$$\|x_{n+1} - q\| \leq \|x_n - x^*\| + k^n_{(1)},$$

where $\sum_{n=1}^{\infty} k^n_{(1)} < \infty$ for all $n \geq 1$. This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + k^n_{(1)}$ and so, by Lemma 1.2, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 2.2, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies *condition*, we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} k^n_{(1)} < \infty$, given any $\epsilon < 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \frac{\epsilon}{4}$ and $\sum_{i=n_0}^{n} k^i_{(1)} < \frac{\epsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F(T)$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{4}$. For $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &= \|x_{n+m} - y^*\| + \|x_n - y^*\| \\ &\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{i=n_0}^{n} k^i_{(1)} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. Then $d(u, F(T)) = 0$. It follows that $u \in F(T)$. As in the proof of Theorem 2.3, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

it follows that $\lim_{n\to\infty} y_n = u$. This completes the proof. □

If $a_n = \mu_n = \delta_n \equiv 0$, then the iterative scheme (1.2) reduces to that of (1.3) and the following result is directly obtained by Theorem 2.4.

Theorem 2.5. *(Shahzad [12] Theorem 3.6, p.1037). Let X be a real uniformly convex Banach space and C a nonempty closed convex subset of X which is also a nonexpansive retract of X. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (1.3). Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to some fixed point of T.*

In the next result, we prove the weak convergence of the new iterative scheme (1.2) for nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} b_n$, and $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.2) converge weakly to a fixed point of T.*

Proof. By using the same proof as in Lemma 2.2, it can be shown that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.5, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.5, $u, v \in F(T)$. By Lemma 1.2, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.6 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. As in the proof of Theorem 2.3, we havelim$_{n\to\infty} \|y_n - x_n\| = 0$ and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ weakly as $n \to \infty$. The proof is completed. □

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