GENERALIZED SEQUENTIALLY COHEN-MACAULAY MODULES UNDER BASE CHANGE

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Abstract

Assumethat $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ isalocalflathomomorphismbetweencommutativeNoetherian localrings R and S.Let M beafinitelygeneralized $R-$module.Theascentanddescentofgeneralized sequentiallyCohen-Macaulaynessbetween $R-$module M and $S-$module $M \otimes_R S$ aregiven.Anexample isgiventopointoutthattheresultofM.TousiandS.Yassemi[10]cannotbeextendedforgeneralized sequentiallyCohen-Macaulaymodules.

Key words: GeneralizedsequentiallyCohen-Macaulaymodules,flathomomorphisms, $f-$sequences.

1 Introduction

Throughout this paper, $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ is a local flat homomorphism between commutative Noetherian local rings R and S. Let M be a non-zero finitely generated $R-$module with $\dim M = d$. It is well known that the studying properties of modules via a local flat homomorphism is an extremely useful technique in commutative algebra. For example, cf. [2], it is proved that S is a complete intersection ring (rep. Gorenstein ring, Cohen-Macaulay ring) if and only if R and $S/\mathfrak{m}S$ are complete intersection (rep. Gorenstein, Cohen-Macaulay). Moreover, if S is regular then so is R, and conversly, if R and $S/\mathfrak{m}S$ are regular then so is S. Recently, M. Tousi and S. Yassemi [10] pointed out the ascent and descent of the sequentially Cohen-Macalayness between $R-$module M and $S-$module $M \otimes_R S$. Concretely, their main theorem (See [10], Theorem 5) gives an equivalence of three following statements:

(i) M is sequentially Cohen-Macaulay $R-$module and $S/\mathfrak{m}S$ is Cohen-Macaulay ring;

(ii) $M \otimes_R S$ is sequentially Cohen-Macaulay $S-$module and

$$0 = M_0 \otimes_R S \subset M_1 \otimes_R S \subset \cdots \subset M_t \otimes_R S = M \otimes_R S$$

is a dimension filtration of $M \otimes_R S$;

(iii) $M \otimes_R S$ is sequentially Cohen-Macaulay $S-$module and $\operatorname{Ass}_S(S/\mathfrak{p}S) = \operatorname{Ass}_S^{k+\ell}(S/\mathfrak{p}S)$ for each $\mathfrak{p} \in \operatorname{Ass}_R^k(M)$ and each $k = 0, 1, \cdots, d-1$, where, for each $0 \leqslant i \leqslant d$,

$$\operatorname{Ass}_R^i(M) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) | \dim(R/\mathfrak{p}) > i\},$$

and $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ is a dimension filtration of M (i.e a filtration of submodules of M such that M_{i-1} is the largest submodule of M_i which has dimension strictly less than $\dim M_i$ for all $i = 1, \cdots, t$).

Recall that the concept of sequentially Cohen-Macaulay module was introduced by Stanley [8] for graded modules and studied further by Herzog and Sbarra [6]. After that, in [4] N. T. Cuong and L. T. Nhan defined this notion for modules over local rings as follows: An $R-$module

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M is called *sequentially Cohen-Macaulay module* if there exists a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_t = M$ of submodule of M such that

(i) Each quotient N_i/N_{i-1} is Cohen-Macaulay;

(ii) $\dim N_1/N_0 < \dim N_2/N_1 < \cdots < \dim N_t/N_{t-1}$.

They also defined the notion of generalized sequentially Cohen-Macaulay modules which is similar to that of sequentially Cohen-Macaulay modules, except the condition (i) to be replated by the generalized Cohen-Macaulayness of modules N_i/N_{i-1}.

It is natural to ask whether the above results of M. Tousi and S. Yassemi ([10], Theorem 5) can be extended to generalized sequentially Cohen-Macaulay modules ? The aim of this paper is to give an answer to this question. The main result of this paper is as follows

Theorem. *Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism and let $\dim S/\mathfrak{m}S = \ell$. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a dimension filtration of M, $\dim M = d > 0$. Then we have*

(i) If $M \otimes_R S$ is generalized sequentially Cohen-Macaulay $S-$module and $0 = M_0 \otimes_R S \subset M_1 \otimes_R S \subset \cdots \subset M_t \otimes_R S = M \otimes_R S$ is a dimension filtration of $M \otimes_R S$ then M is generalized sequentially Cohen-Macaulay $R-$module. Furthermore, if $\dim S/\mathfrak{m}S > 0$ then M is sequentially Cohen-Macaulay $R-$module.

(ii) If $M \otimes_R S$ is generalized sequentially Cohen-Macaulay $S-$module and

$$\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Ass}_S^{k+\ell}(S/\mathfrak{p}S), \forall\, \mathfrak{p} \in \mathrm{Ass}_R^k(M)$$

for every $k = 1, \cdots, d-1$ then M is generalized sequentially Cohen-Macaulay $S-$module. Furthermore, if $\dim S/\mathfrak{m}S > 0$ then M is sequentially Cohen-Macaulay $R-$module.

We also present an example to show that the result of M. Tousi and S. Yassemi in general can not be extended for generalized sequentially Cohen-Macaulay (see Section 3).

2 Proof of Theorem

To prove the Theorem, we need a result on the generalized Cohen-Maulayness under base change. Firstly, we recall the concepts of filter regular sequence ($f-$sequence) and $f-$module introduced by N. T. Cuong, P. Schenzel and N. V. Trung [5]: a sequence of elements $x_1, \cdots, x_n$ of $\mathfrak{m}$ is called *filter regular sequence* ($f-$*sequence*) with respect to M if

$$(x_1, ..., x_{i-1})M \underset{M}{:} x_i \subseteq \bigcup_{t \geqslant 0} (x_1, ..., x_{i-1})M \underset{M}{:} \mathfrak{m}^t$$

for $i = 1, \cdots, n$, where we stipulate, when $i = 1$ then

$$0 \underset{M}{:} x_1 \subseteq \bigcup_{t \geqslant 0} (0 \underset{M}{:} \mathfrak{m}^t).$$

We say that $R-$module M is $f-$*module* if every system of parameters of M is $f-$sequence. In general, a generalized Cohen-Macaulay module is an $f-$module and the inverse is true when R is an epimorphic image of a local Cohen-Macaulay ring ([9], Appendix, Proposition 16).

Lemma 2.1. *Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism, M a finitely generated $R-$module. Then, if $M \otimes_R S$ is a generalized Cohen-Macaulay $S-$module then M is a generalized Cohen-Macaulay $R-$module. Further, if $\dim S/\mathfrak{m}S > 0$ then M is Cohen-Macaulay module.*

Proof of Lemma 2.1. Let $\widehat{R}$ and $\widehat{S}$ are $\mathfrak{m}-$adic completion of R and S respectively. We have the natural flat local homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\mathfrak{m}}) \longrightarrow (\widehat{S}, \widehat{\mathfrak{n}})$ and $\widehat{S}/\widehat{\mathfrak{m}} \cong \widehat{(S/\mathfrak{m})}$. Hence, without loss of the generality we can suppose that R and S are complete. Therefore they are homomorphic images of regular rings. And then, according to the note above we only have to prove M is $f-$module. Let $(x_1, ..., x_d)$ be any system of parameters of M. Because the exactness of following sequence

$$0 \longrightarrow (x_1, ..., x_d)M \longrightarrow M \longrightarrow M/(x_1, ..., x_d)M \longrightarrow 0.$$

and S is flat as $R-$module, we have exact sequence

$$0 \longrightarrow ((x_1, ..., x_d)M) \otimes_R S \longrightarrow M \otimes_R S \longrightarrow (M/(x_1, ..., x_d)M) \otimes_R S \longrightarrow 0.$$

On the other hand, since S is flat and $(x_1, ..., x_d)$ is finitely generated,

$$((x_1, ..., x_d)M) \otimes_R S \cong (x_1, ..., x_d)(M \otimes_R S),$$

So that

$$(M/(x_1, ..., x_d)M) \otimes_R S \cong (M \otimes_R S)/(x_1, ..., x_d)(M \otimes_R S).$$

It follows

$$\begin{aligned} \dim(M \otimes_R S)/(x_1, ..., x_d)(M \otimes_R S) &= \dim((M/(x_1, ..., x_d)M) \otimes_R S) \\ &= \dim M/(x_1, ..., x_d)M + \dim S/\mathfrak{m}S \\ &= \dim S/\mathfrak{m}S \\ &= \dim(M \otimes_R S) - \dim M. \end{aligned}$$

That proved $(x_1, ..., x_d)$ is a part of system of parameters of $M \otimes_R S$. According to the hypothesis of the lemma, $M \otimes_R S$ is generalized Cohen-Macaulay $S-$module then $(x_1, ..., x_d)$ is $f-$sequence of $M \otimes_R S$. It follows

$$\ell\left(0 :_{(M \otimes_R S)/(x_1, ..., x_{i-1})(M \otimes_R S)} x_i\right) < \infty,$$

for all $i = 1, ..., d$. It is easy to verify that

$$\begin{aligned} \left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) \otimes_R S &\cong 0 :_{(M/(x_1, ..., x_{i-1})M) \otimes_R S} x_i \\ &\cong 0 :_{(M \otimes_R S)/(x_1, ..., x_{i-1})(M \otimes_R S)} x_i. \end{aligned}$$

Then $\ell\left(\left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) \otimes_R S\right) < \infty$, or $\dim\left(\left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) \otimes_R S\right) \leqslant 0$. It follows that

$$\dim\left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) + \dim S/\mathfrak{m}S \leqslant 0.$$

We have $\dim S/\mathfrak{m}S \geqslant 0$ because $\varphi : R \longrightarrow S$ is local and according to Nakayama's lemma. So that

If $\dim S/\mathfrak{m}S = 0$ then

$$\dim\left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) \leqslant 0,$$

and we have M is $f-$module.

If $\dim S/\mathfrak{m}S > 0$ then

$$\dim\left(0 :_{M/(x_1, ..., x_{i-1})M} x_i\right) < 0$$

or $0 :_{M/(x_1, ..., x_{i-1})M} x_i = 0$, for all i. This equivalent that $(x_1, ..., x_d)$ is regular sequence of M and implies M is Cohen-Macaulay $R-$module. □

Proof of Theorem . According to ([10], Lemma 4 (a)) we only have to prove (i). It follow by ([4], Lemma 4.4 (iii)) $M_i \otimes_R S/M_{i-1} \otimes_R S$ is generalized Cohen-Macaulay. From the following isomorphism

$$(M_i \otimes_R S)/(M_{i-1} \otimes_R S) \cong (M_i/M_{i-1}) \otimes_R S$$

and the above lemma we have the requirement. □

3 Example

Before giving an example to point out that the result of M. Tousi and S. Yassemi can not be extended for generalized sequentially Cohen-Macaulay modules, we give a result about the polynomial type of the ring of the formal series power and the polynomial ring. Recall that the notion *polynomial type* of modules was introduced by N. T. Cuong in [3] as follows. For a system of parameters $\underline{x} = (x_1, ..., x_d)$ of M and a set of positive integers $\underline{n} = (n_1, ..., n_d)$, we set $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$. Consider the difference

$$I(\underline{x}(\underline{n}); M) = \ell(M/\underline{x}(\underline{n})M) - n_1...n_d e(\underline{x}; M)$$

as a function in $n_1, ..., n_d$, where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$. In general, $I(\underline{x}(\underline{n}); M)$ is not polynomial for $n_1, ..., n_d$ large enough, but they are still nice since they are bounded above by polynomials. Especially, the least degree of all polynomials in $\underline{n}$ bounding above $I(\underline{x}(\underline{n}); M)$ is independent of the choice of $\underline{x}$, and it is denoted by $p(M)$. The invariant $p(M)$ is called the *polynomial type* of M. If we stipulate the degree of the zero polynomial is $-\infty$, then M is a Cohen-Macaulay module if and only if $p(M) = -\infty$, and M is generalized Cohen-Macaulay module if and only if $p(M) \leqslant 0$.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Then*

(i) $p(R[[X_1, ..., X_n]]) = p(R) + n$.

(ii) $p(R[X_1, ..., X_n]_{(\mathfrak{m}, X_1, ..., X_n)R[X_1, ..., X_n]}) = p(R) + n$,
where $R[[X_1, ..., X_n]]$ and $R[X_1, ..., X_n]$ are respectively the ring of formal power series and the polynomial ring of variables $X_1, \cdots, X_n$.

Proof. We only need to prove the case $n = 1$. Set $T = R[[X]], S = R[X]_{(\mathfrak{m},X)R[X]}$.

(i) Let $(x_1, ..., x_t)$ be a system of parameter if R. It is obvious that $(x_1, ..., x_t, X)$ is system of parameter of T. Because of the regularity of X, we have

$$\begin{aligned} \ell_T(T/(x_1^{n_1}, ..., x_t^{n_t}, X^m)T) &= m\ell_T(T/(x_1^{n_1}, ..., x_t^{n_t}, X)T) \\ &= m\ell_{T/X}(T/X/(x_1^{n_1}, ..., x_t^{n_t}, X)T/X) \\ &= \ell_R(R/(x_1^{n_1}, ..., x_t^{n_t})R). \end{aligned}$$

and

$$\begin{aligned} e_T(x_1^{n_1}, ..., x_t^{n_t}, X^m; T) &= me_T(x_1^{n_1}, ..., x_t^{n_t}, X; T) \\ &= me_{T/X}(x_1^{n_1}, ..., x_t^{n_t}, X; T/X) \\ &= me_R(x_1^{n_1}, ..., x_t^{n_t}; R). \end{aligned}$$

So that

$$I(x_1^{n_1}, ..., x_t^{n_t}, X^m; T) = mI(x_1^{n_1}, ..., x_t^{n_t}; R)$$

or $p(T) = p(R) + 1$, where we denote

$$I(\underline{x}(\underline{n}); M) = \ell(M/\underline{x}(\underline{n})M) - n_1...n_d e(\underline{x}; M),$$

for $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$.

(ii) Set $\varphi = gf$ is natural homomorphism from R to S

$$R \xrightarrow{f} R[z] \xrightarrow{g} S.$$

Let $(x_1, ..., x_t)$ be a system of parameters of R. By ([7], 7.8), we have $(g(x_1), ..., g(x_t), g(X))$ is a system of parameters of S and

$$(R[X]/(x_1^{n_1}, ..., x_t^{n_t}, X^m)R[X])_{(\mathfrak{m},X)R[X]} \cong S/(g(x_1)^{n_1}, ..., g(x_t)^{n_t}, g(X)^m)S.$$

From that and ([7], 3.9 Theorem 12), we have

$$\begin{aligned}\ell_S(S/(g(x_1)^{n_1}, ..., g(x_t)^{n_t}, g(X)^m)S) &= \ell_{R[X]}(R[X]/(x_1^{n_1}, ..., x_t^{n_t}, X^m)R[X]) \\ &= m\ell_R(R/(x_1^{n_1}, ..., x_t^{n_t})R).\end{aligned}$$

Also by ([7], 7.8 Theorem 15) we have

$$e_S(g(x_1)^{n_1}, ..., g(x_t)^{n_t}, g(X)^m; S) = me_R(x_1^{n_1}, ..., x_t^{n_t}; R).$$

It follows that

$$I(g(x_1)^{n_1}, ..., g(x_t)^{n_t}, g(X)^m; S) = mI(x_1^{n_1}, ..., x_t^{n_t}; R).$$

or $p(S) = p(R) + 1$. □

Proposition 3.2. *Let R be a local domain with maximal ideal $\mathfrak{m}$. R is generalized sequentially Cohen-Macaulay but is not sequentially Cohen-Macaulay. Set $S = R[X]_{(\mathfrak{m},X)R[X]}$, where X is variable on R. The following statements are true*

(i) The natural homomorphism $\varphi : R \longrightarrow S$ is flat local homomorphism.

(ii) R is generalized sequentially Cohen-Macaulay $R-$module but $R \otimes_R S$, as a $S-$module, is not.

Proof. (i) It is obvious.

(ii) Since R is domain, R is generalized sequentially Cohen-Macaulay with a dimension filtration $0 \subset R$. So that R is generalized Cohen-Macaulay. We also have $R \otimes_R S = S$ is a domain, following $0 \subset R \otimes_R S$ is a dimension filtration of $R \otimes_R S$. Assume that $R \otimes_R S$ is generalized sequentially Cohen-Macaulay then $R \otimes_R S$ is generalized Cohen-Macaulay. According to [3.1, (ii)], we have

$$p(R \otimes_R S) = p(S) = p(R) + 1 = 1,$$

contradiction. Therefore $R \otimes_R S$ is not generalized sequentially Cohen-Macaulay. □

Example 3.1. *Let k be any field, $k[[x, y]]$ a ring of formal series with variables x and y. Set $R = k[[x^4, x^3y, xy^3, y^4]]$ in $k[[x, y]]$. It is easily to verify that R is domain, R is generalized Cohen-Macaulay but is not Cohen-Macaulay. So that R is generalized sequentially Cohen-Macaulay but is not sequentially Cohen-Macaulay. According to [3.2] we have a example satisfy the requirement. Note that, in this case $S/\mathfrak{m}S$ is Cohen-Macaulay, where $S = R[X]_{(\mathfrak{m},X)R[X]}$, X is variable on R and $\mathfrak{m}$ is maximal ideal of R.*

Acknowledgment. This article is content of Chapter 3 in [1].

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TÓM TẮT

MÔĐUN COHEN-MACAULAY SUY RỘNG QUA BIẾN ĐỔI CƠ SỞ

Giả sử $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ là một đồng cấu phẳng, địa phương giữa các vành địa phương, Noether, giao hoán R và S. Giả sử M là một $R-$môđun hữu hạn sinh. Tính tăng giảm Cohen-Macaulay suy rộng dãy giữa $R-$môđun M và $S-$môđun $M \otimes_R S$ được nghiên cứu trong bài báo. Một ví dụ được đưa ra trong bài báo chứng tỏ kết quả của M. Tousi và S. Yassemi [10] không mở rộng được cho trường hợp Cohen-Macaulay suy rộng dãy.

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Từ khoá: Đồng cấu phẳng; Môđun Cohen-Macaulay suy rộng dãy, f-dãy .

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