Fixed point theorem using a contractive condition of rational expression in the context of ordered partial metric spaces

Nguyen Thanh Mai
University of Science, Thainguyen University, Vietnam
E-mail: thanhmai6759@gmail.com

Abstract The purpose of this manuscript is to present a fixed point theorem using a contractive condition of rational expression in the context of ordered partial metric spaces.

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1 Introduction and preliminaries

Partial metric is one of the generalizations of metric was introduced by Matthews[2] in 1992 to study denotational semantics of data flow networks. In fact, partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory [1, 4, 6, 7, 8, 11]. Recently, many researchers have obtained fixed, common fixed and coupled fixed point results on partial metric spaces and ordered partial metric spaces [3, 5, 6, 9, 10]. In [12] Harjani et al. proved the following fixed point theorem in partially ordered metric spaces.

Theorem 1.1. ([12]). Let $(X, \leq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y) \text{ for } x, y \in X, x \geq y, x \neq y,$$

Also, assume either T is continuous or X has the property that (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

In this paper we extend the result of Harjani, Lopez and Sadarangani [12] to the case of partial metric spaces. An example is considered to illustrate our obtained results. First, we recall some definitions of partial metric space and some of their properties [2, 3, 4, 5, 10].

Definition 1.2. *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:*

(P1) $p(x, y) = p(y, x)$ *(symmetry);*

(P2) *if* $0 \leq p(x, x) = p(x, y) = p(y, y)$ *then* $x = y$ *(equality);*

(P3) $p(x,x) \le p(x,y)$ *(small self-distances);*

(P4) $p(x,z) + p(y,y) \le p(x,y) + p(y,z)$ *(triangularity); for all* $x, y, z \in X$.

For a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}_+$ given by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Lemma 1.3. *Let* (X,p) *be a partial metric space. Then*

(i) *A sequence* $\{x_n\}$ *is a Cauchy sequence in the PMS* (X,p) *if and only if* $\{x_n\}$ *is Cauchy in a metric space* (X,d_p).

(ii) *A PMS* (X,p) *is complete if and only if a metric space* (X,d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

2 Main Results

Theorem 2.1. *Let* $(X,\le)$ *be a partially ordered set and suppose that there exists a partial metric p in* X *such that* (X,p) *is a complete partial metric space. Let* $T : X \to X$ *be a continuous and nondecreasing mapping such that*

$$p(Tx,Ty) \le \frac{\alpha p(x,Tx)p(y,Ty)}{p(x,y)} + \beta p(x,y), \ \text{for } x,y \in X, x \ge y, x \ne y, \qquad (1)$$

with $\alpha > 0, \beta > 0, \alpha + \beta < 1$. *If there exists* $x_0 \in X$ *with* $x_0 \le Tx_0$, *then* T *has fixed point* $z \in X$ *and* $p(z,z) = 0$.

Proof. If $Tx_0 = x_0$, then the proof is done. Suppose that $x_0 \le Tx_0$.
Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 \le Tx_0 \le T^2x_0 \le \cdots \le T^nx_0 \le T^{n+1}x_0 \le \cdots$$

Put $x_{n+1} = Tx_n$. If there exists $n \ge 1$ such that $x_{n+1} = x_n$, then from $x_{n+1} = Tx_n = x_n$, x_n is a fixed point. Suppose that $x_{n+1} \ne x_n$ for $n \ge 1$. That is x_n and x_{n-1} are comparable, we get, for $n \ge 1$,

$$\begin{aligned} p(x_{n+1},x_n) &= p(Tx_n, Tx_{n-1}) \\ &\le \frac{\alpha p(x_n,Tx_n)p(x_{n-1},Tx_{n-1})}{p(x_n,x_{n-1})} + \beta p(x_n,x_{n-1}) \\ &\le \alpha p(x_n,x_{n+1}) + \beta p(x_n,x_{n-1}) \end{aligned}$$

The last inequality gives us

$$\begin{aligned} p(x_{n+1},x_n) &\le kp(x_n,x_{n-1}), \ \ k = \frac{\beta}{1-\alpha} < 1 \\ &\cdots \\ &\le k^n p(x_1,x_0). \end{aligned}$$

$$p(x_{n+1}, x_n) \leq k^n p(x_1, x_0). \tag{2}$$

Moreover, by the triangular inequality, we have, for $m \geq n$,

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + \cdots + p(x_{n+1}, x_n) - \sum_{i=1}^{m-n-1} p(x_{m-k}, x_{m-k}) \\ &\leq [k^{m-1} + \cdots + k^n] p(x_1, x_0) \\ &= k^n \frac{1 - k^{m-n}}{1-k} p(x_1, x_0) \\ &\leq \frac{k^n}{1-k} p(x_1, x_0). \end{aligned}$$

Hence, $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in (X, p). By Lemma 1.3, $\{x_n\}$ is also Cauchy in (X, d_p). In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $z \in X$ such that $x_n \to z$ in (X, d_p); moreover, by Lemma 1.3,

$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Given that T is continuous in (X, p). Therefore, $Tx_n \to Tz$ in (X, p).

$$\text{i.e.,}\ \ p(Tz, Tz) = \lim_{n\to\infty} p(Tz, Tx_n) = \lim_{n,m\to\infty} p(Tx_n, Tx_m).$$

But, $p(Tz, Tz) = \lim_{n,m\to\infty} p(Tx_n, Tx_m) = \lim_{n,m\to\infty} p(x_{n+1}, x_{m+1}) = 0$.
We will show next that z is the fixed point of T.

$$p(Tz, z) \leq p(Tz, Tx_n) + p(Tx_n, z) - p(Tx_n, Tx_n).$$

As $n \to \infty$, we obtain $p(Tz, z) \leq 0$. Thus, $p(Tz, z) = 0$.
Hence $p(z, z) = p(Tz, Tz) = p(Tz, z) = 0$. Therefore, by (P2) we get $Tz = z$ and $p(z, z) = 0$ which completes the proof.

In what follows we prove that Theorem 2.1 is still valid for T not necessarily continuous, assuming X has the property that

$$(x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x, \text{then } x = \sup\{x_n\}. \tag{3}$$

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X, p) is a complete partial metric space. Assume that X satisfies (3) in (X, p). Let $T : X \to X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \leq \frac{\alpha p(x, Tx) p(y, Ty)}{p(x, y)} + \beta p(x, y),\ \textit{for}\ x, y \in X, x \geq y, x \neq y, \tag{4}$$

with $\alpha > 0, \beta > 0, \alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has fixed point $z \in X$ and $p(z, z) = 0$.

Proof. Following the proof of Theorem 2.1, we only have to check that $Tz = z$. As (x_n) is a nondecreasing sequence in X and $x_n \to z$, then, by (3), $z = \sup\{x_n\}$. In particularly, $x_n \leq z$ for all $n \in \mathbb{N}$.
Since T is a nondecreasing mapping, then $Tx_n \leq Tz$, for all $n \in \mathbb{N}$ or, equivalently,

$x_{n+1} \leq Tz$ for all $n \in \mathbb{N}$. Moreover, as $x_0 < x_1 \leq Tz$ and $z = \sup\{x_n\}$, we get $z \leq Tz$. Suppose that $z < Tz$. Using a similar argument that in the proof of Theorem 2.1 for $x_0 \leq Tx_0$, we obtain that $\{T^n z\}$ is a nondecreasing sequence such that

$$p(y, y) = \lim_{n\to\infty} p(T^n z, y) = \lim_{m,n\to\infty} p(T^n z, T^m z) = 0 \text{ for some } y \in X. \tag{5}$$

By the assumption of (3) , we have $y = \sup\{T^n z\}$.
Moreover, from $x_0 \leq z$, we get $x_n = T^n x_0 \leq T^n z$ for $n \geq 1$ and $x_n < T^n z$ for $n \geq 1$ because $x_n \leq z < Tz \leq T^n z$ for $n \geq 1$.
As x_n and $T^n z$ are comparable and distinct for $n \geq 1$, applying the contractive condition we get

$$\begin{aligned} p(T^{n+1}z, x_{n+1}) &= p(T(T^n z), Tx_n) \\ &\leq \frac{\alpha p(T^n z, T^{n+1}z) p(x_n, Tx_n)}{p(T^n z, x_n)} + \beta p(T^n z, x_n), \end{aligned}$$

$$p(T^{n+1}z, x_{n+1}) \leq \frac{\alpha p(T^n z, T^{n+1}z) p(x_n, x_{n+1})}{p(T^n z, x_n)} + \beta p(T^n z, x_n). \tag{6}$$

From $\lim_{n\to\infty} p(x_n, z) = \lim_{n\to\infty} p(T^n z, y) = 0$, we have

$$\lim_{n\to\infty} p(T^n z, x_n) = p(y, z). \tag{7}$$

As, $n \to \infty$ in (6) and using that (2) and (7), we obtain

$$p(y, z) \leq \beta p(y, z).$$

As $\beta < 1$, $p(y, z) = 0$. Hence $p(z, z) = p(y, y) = p(y, z) = 0$. Therefore, by (P2) $y = z$. Particularly, $y = z = \sup\{T^n z\}$ and, consequently, $Tz \leq z$ and this is a contradiction. Hence, we conclude that $z = Tz$ and $p(z, z) = 0$.

Example 2.3. *Let $X = [0, \infty)$ endowed with the usual partial metric p defined by $p : X \times X \to \mathbb{R}_+$ with $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. We consider the ordered relation in X as follows*

$$x \preccurlyeq y \Leftrightarrow p(x, x) = p(x, y) \Leftrightarrow x = \max\{x, y\} \Leftarrow y \leq x$$

where $\leq$ be the usual ordering.

It is clear that $(X, \preccurlyeq)$ is totally ordered. The partial metric space (X, p) is complete because (X, d_p) is complete. Indeed, for any $x, y \in X$,

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - (x + y) = |x - y|$$

Thus, $(X, d_p) = ([0, \infty), |.|)$ is the usual metric space, which is complete.
Let $T : X \to X$ be given by $T(x) = \dfrac{x}{2}$, $x \geq 0$. The function T is continuous on (X, p). Indeed, let $\{x_n\}$ be a sequence converging to x in (X, p), then $\lim_{n\to\infty} \max\{x_n, x\} = \lim_{n\to\infty} p(x_n, x) = p(x, x) = x$ and

$$\lim_{n\to\infty} p(Tx_n, Tx) = \lim_{n\to\infty} \max\{Tx_n, Tx\} = \lim_{n\to\infty} \frac{\max\{x_n, x\}}{2} = \frac{x}{2} = p(Tx, Tx) \tag{8}$$

that is $\{T(x_n)\}$ converges to $T(x)$ in (X,p). Since $x_n \to x$ and by the definition T we have, $\lim_{n\to\infty} d_p(x_n, x) = 0$ and

$$\lim_{n\to\infty} d_p(Tx_n, Tx) = 0. \tag{9}$$

From (8) and (9) we get T is continuous on (X,p). Any $x, y \in X$ are comparable, so for example we take $x \preccurlyeq y$, and then $p(x,x) = p(x,y)$, so $y \leq x$. Since $T(y) \leq T(x)$, so $T(x) \preccurlyeq T(y)$, giving that T is non-decreasing with respect to $\preccurlyeq$. In particular, for any $x \preccurlyeq y$, we have

$$p(x,y) = x, p(Tx,Ty) = Tx = \frac{x}{2}, p(x,Tx) = x, p(y,Ty) = y.$$

Now we have to show that T satisfies the inequality (1) For any $x, y \in X$ with $x \preccurlyeq y$ and $x \neq y$, we have

$p(Tx,Ty) = \dfrac{x}{2}$ and $\dfrac{\alpha p(x,Tx)p(y,Ty)}{p(x,y)} + \beta p(x,y) = \dfrac{\alpha xy}{x} + \beta x.$

Therefore, choose $\beta \geq \frac{1}{2}$ and $\alpha + \beta < 1$, then (1) holds. All the hypotheses are satisfied, so T has a unique fixed point in X which is 0 and $p(0,0) = 0$.

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Tóm tắt

Định lý điểm bất động sử dụng một điều kiện co
trong không gian metric được sắp thứ tự bộ phận

Nguyễn Thanh Mai

Trường Đại học Khoa học - Đại học Thái Nguyên

Bài báo này giới thiệu định lý điểm bất động sử dụng một điều kiện co trong không gian metric được sắp thứ tự bộ phận.

Từ khoá: Không gian metric, điểm bất động, tập có thứ tự.