A FINITENESS RESULT FOR ASSOCIATED PRIMES OF CERTAIN EXT-MODULES

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Abstract [1]. Using some properties of unconditioned $M-$sequences in dimension $> s$, we give a finiteness result for the set $\bigcup_{n\in\mathbb{N}} \mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I^n, M))$.

1 Introduction

Throughout this paper, let R be a Noetherian commutative ring, let M be a finitely generated $R-$module, and A an Artinian $R-$module.

For an ideal I of R, it was shown in [B] that the two sequences of associated primes

$$\mathrm{Ass}_R(M/I^nM) \text{ and } \mathrm{Ass}_R(I^nM/I^{n+1}M), n = 1, 2, \ldots$$

eventually become constant for large n. Sharp [Sh] proved the dual result for Artinian modules: $\mathrm{Att}_R(0:_A I^n)$ and $\mathrm{Att}_R\big((0:_A I^n)/(0:_A I^{n-1})\big)$ do not depend on n for n large. Starting from the observation that $M/I^nM \cong \mathrm{Tor}^R_0(R/I^n, M)$ and $0:_A I^n \cong \mathrm{Ext}^0_R(R/I^n, A)$ for any n, Melkersson and Schenzel [MS] extended the above results as follows: For any given integer $i \geq 0$, the sequences

$$\mathrm{Ass}_R\big(\mathrm{Tor}^R_i(R/I^n, M)\big) \text{ and } \mathrm{Att}_R\big(\mathrm{Ext}^i_R(R/I^n, A)\big), n = 1, 2, \ldots$$

become independent of n for large n. Melkersson and Schenzel [MS] also asked whether the set $\mathrm{Ass}_R\big(\mathrm{Ext}^i_R(R/I^n, M)\big)$ is independent of n for large n.

In fact, $\bigcup_{n\in\mathbb{N}} \mathrm{Ass}_R\big(\mathrm{Ext}^i_R(R/I^n, M)\big)$ is not a finite set in general, and therefore the set $\mathrm{Ass}_R\big(\mathrm{Ext}^i_R(R/I^n, M)\big)$ depends on n for n large. Indeed, Katzman [Ka, Corollary 1.3] gave an example of a Noetherian local ring $(R, \mathfrak{m})$ with two elements $x, y \in \mathfrak{m}$ such that $\mathrm{Ass}_R\big(H^2_{(x,y)R}(R)\big)$ is an infinite set. Therefore the set $\bigcup_{n\in\mathbb{N}} \mathrm{Ass}_R\big(\mathrm{Ext}^2_R(R/(x,y)^n, R)\big)$ is infinite.

For convenience, for a subset T of $\mathrm{Spec}\,R$ and an integer $i \geq 0$, we set

$$(T)_i := \{\mathfrak{p} \in T \ : \ \dim R/\mathfrak{p} = i\};$$
$$(T)_{\geq i} := \{\mathfrak{p} \in T \ : \ \dim R/\mathfrak{p} \geq i\}.$$

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For an integer $i \geq 0$, an ideal I of R, and a system $\underline{a} = (a_1, \ldots, a_k)$ of elements in R, we set

$$T^i(I, M) := \bigcup_{n \in \mathbb{N}} \operatorname{Ass}_R\left(\operatorname{Ext}^i_R(R/I^n, M)\right);$$
$$T^i(\underline{a}, M) := \bigcup_{n_1, \ldots, n_k \in \mathbb{N}} \operatorname{Ass}_R\left(\operatorname{Ext}^i_R(R/(a_1^{n_1}, \ldots, a_k^{n_k}), M)\right).$$

In this paper, we prove the following finiteness result for the sets $T^i(I, M)$ and $T^i(\underline{a}, M)$.

Theorem 1.1. *Let $s \geq 0$ and $r \geq 1$ be integers. Assume that $\dim(\operatorname{Supp}(H^i_I(M))) \leq s$ for all $i < r$. Then for any system of generators $\underline{a} = (a_1, \ldots, a_k)$ of I and for all integers $t \leq r$, the sets $\Big(T^t(I, M)\Big)_{\geq s}$ and $\Big(T^t(\underline{a}, M)\Big)_{\geq s}$ are contained in the finite set $\bigcup_{i=0}^{t} \operatorname{Ass}_R \operatorname{Ext}^i_R(R/I, M)$.*

Theorem 1.2. *Let $s \geq 0$ and $r \geq 1$ be integers. Assume that $\dim(\operatorname{Supp}(H^i_I(M))) \leq s$ for all $i < r$. Let $x_1, \ldots, x_r \in I$ be a sequence which is at the same time an unconditioned $M-$sequence in dimension $> s$ and an unconditioned $I-$filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any system of generators $\underline{a} = (a_1, \ldots, a_k)$ of I and for all integers $t \leq r$, the sets $\Big(T^t(I, M)\Big)_{\geq s}$ and $\Big(T^t(\underline{a}, M)\Big)_{\geq s}$ are contained in the finite set*

$$\Big(\operatorname{Ass}_R(M/(x_1, \ldots, x_t)M)\Big)_{\geq s+1} \;\cup\; \Big(\bigcup_{i=0}^{t} \operatorname{Ass}_R(M/(x_1, \ldots, x_i)M)\Big)_s.$$

2 Unconditioned $M-$sequences in dimension $> s$

Definition 2.1. Let $s \geq 0$ be an integer, let $x_1, \ldots, x_r \in R$ be a sequence. We say that $x_1, \ldots, x_r$ is an *$M-$sequence in dimension $> s$* if $x_1, \ldots, x_r$ is a poor $M_{\mathfrak{p}}-$sequence for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim(R/\mathfrak{p}) > s$.

Observe that $x_1, \ldots, x_r$ is an $M-$sequence in dimension $> s$ if and only if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \Big(\operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M)\Big)_{\geq s+1}$ and all $i = 1, \ldots, r$.

Assume that R is local. Then $x_1, \ldots, x_r$ is an $M-$sequence in dimension > 0 if and only if it is a filter regular sequence with respect to M in sense of [Cst]. Moreover, $x_1, \ldots, x_r$ is an $M-$sequence in dimension > 1 if and only if it is a generalized regular sequence with respect to M in sense of [Nh].

Reminder 2.2. (a) Let I be an ideal. A sequence $x_1, \ldots, x_r \in I$ is called an *$I-$filter regular sequence* with respect to M if $x_1, \ldots, x_r$ is an $M_{\mathfrak{p}}-$sequence for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(I)$. It is equivalent to say that $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M) \setminus \operatorname{Var}(I)$ and all $i = 1, \ldots, r$.

(b) (cf. [NS, 3.4], [Kh, 2.1]). If $x_1, \dots, x_r$ is an $I-$filter regular sequence with respect to M then

$$H_I^j(M) = \begin{cases} H^j_{(x_1,\dots,x_r)R}(M), & \text{if } j < r \\ H_I^{j-r}(H^r_{(x_1,\dots,x_r)R}(M)), & \text{if } j \geq r. \end{cases}$$

Definition 2.3. A sequence $x_1, \dots, x_r \in R$ is called an *unconditioned $M-$sequence in dimension $> s$* if $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is an $M-$sequence in dimension $> s$ for all permutations $\sigma \in \mathbb{S}_r$. Similarly, a sequence $x_1, \dots, x_r \in I$ is called an *unconditioned $I-$filter regular sequence* with respect to M if $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is an $I-$filter regular sequence with respect to M for all permutations $\sigma \in \mathbb{S}_r$.

Lemma 2.4. *Let $s \geq 0$ be an integer, let I be an ideal of R.*

(a) Let $r > 0$ be an integer. Then $\dim(\operatorname{Supp}(H_I^i(M))) \leq s$ *for all $i < r$ if and only if there exists an $M-$sequence in dimension $> s$ of length r in I.*

(b) If $\dim(M/IM) > s$ *then each $M-$sequence in dimension $> s$ in I may be extended to a maximal $M-$sequence in dimension $> s$ in I. Moreover, all maximal $M-$sequences in dimension $> s$ in I have the same length, and this common length is equal to the least integer i such that* $\dim(\operatorname{Supp}(H_I^i(M))) > s$.

(c) If $\dim(M/IM) \leq s$ *then there exists an $M-$sequence in dimension $> s$ in I of length n for any integer $n > 0$.*

Proof. (a). Assume that $\dim(\operatorname{Supp}(H_I^i(M))) \leq s$ for all $i < r$. We prove by induction on r that there is a sequence $x_1, \dots, x_r \in I$ which is an $M-$sequence in dimension $> s$. Let $r \geq 1$. Then $\dim(\operatorname{Supp}(H_I^0(M))) \leq s$. Hence $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \left(\operatorname{Ass}_R M\right)_{\geq s+1}$. Therefore there exists an element $x_1 \in I$ which is $M-$regular in dimension $> s$. This proves the case $r = 1$. Let $r > 1$ and set $x_1 = x$. Then $\dim(0 :_M x) \leq s$. From the exact sequence $0 \longrightarrow 0 :_M x \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0$ we get an exact sequence

$$H_I^i(M) \longrightarrow H_I^i(M/(0 :_M x)) \longrightarrow H_I^{i+1}(0 :_M x)$$

for all $i \geq 0$. As $\dim(0 :_M x) \leq s$, we have $\dim(\operatorname{Supp}(H_I^i(0 :_M x))) \leq s$ for all $i \geq 0$. Therefore, by our hypothesis, $\dim(\operatorname{Supp}(H_I^i(M/(0 :_M x))) \leq s$ for all $i < r$. From the exact sequence

$$0 \longrightarrow M/(0 :_M x) \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

we get an exact sequence $H_I^i(M) \longrightarrow H_I^i(M/xM) \longrightarrow H_I^{i+1}(M/(0 :_M x))$ for all $i \geq 0$. So, $\dim(\operatorname{Supp}(H_I^i(M/xM))) \leq s$ for all $i < r-1$. By induction, there exists a sequence $x_2, \dots, x_r$ in I which is an $M/xM-$sequence in dimension $> s$. So, $x_1, \dots, x_r$ is an $M-$sequence in dimension $> s$ in I.

Conversely, assume $x_1, \dots, x_r$ is an $M-$sequence in dimension $> s$ in I. Let $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim(R/\mathfrak{p}) > s$. Then $\frac{x_1}{1}, \dots, \frac{x_r}{1}$ is a poor $M_\mathfrak{p}-$sequence in $I_\mathfrak{p}$. So, $H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$, i.e. $\mathfrak{p} \notin \operatorname{Supp}(H_I^i(M))$ for all $i < r$. Therefore $\dim(\operatorname{Supp} H_I^i(M))) \leq s$ for all $i < r$.

(b). Since $\dim(M/IM) > s$, there is a maximal ideal $\mathfrak{m}$ such that $\dim(M_\mathfrak{m}/IM_\mathfrak{m}) > s$. Note that each $M-$sequence in dimension $> s$ in I is an $M_\mathfrak{m}-$sequence in dimension $> s$ in $IR_\mathfrak{m}$.

As $\dim(M_{\mathfrak{m}}/IM_{\mathfrak{m}}) > s$, each $M_{\mathfrak{m}}-$sequence in dimension $> s$ in $IR_{\mathfrak{m}}$ is a part of a system of parameters of $M_{\mathfrak{m}}$. Therefore the length of an $M-$sequence in dimension $> s$ in I is at most $\dim M_{\mathfrak{m}} - s - 1$. So, there is a common bound on the lengths of all $M-$sequences in dimension $> s$ which consist of elements in I. Therefore, each $M-$sequence in dimension $> s$ in I may be extended to a maximal $M-$sequence in dimension $> s$ in I.

Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_t$ be maximal $M-$sequences in dimension $> s$ in I. Assume that $r \neq t$, say $r < t$. By (a), $\dim(\operatorname{Supp} H_I^i(M))) \leq s$ for all $i \leq r$. Similar as in the proof of (a), it follows by induction on k that $\dim(\operatorname{Supp}(H_I^i(M/(x_1, \ldots, x_k)M))) \leq s$ for all $i \leq r-k$ and all $k \leq r$. Thus $\dim(H_I^0(M/(x_1, \ldots, x_r)M)) \leq s$. Therefore there is an element in I which is $M/(x_1, \ldots, x_r)M-$regular in dimension $> s$. This is a contradiction to the maximality of the sequence $(x_1, \ldots, x_r)$. So, all maximal $M-$sequences in dimension $> s$ in I have the same length and by (a) this length has the stated value.

(c) is clear. $\square$

Proposition 2.5. *Let $s \geq 0$ and $r \geq 1$ be integers, and let $I \subseteq R$ be an ideal. If $\dim(\operatorname{Supp}(H_I^i(M))) \leq s$ for all $i < r$ then there is a sequence $x_1, \ldots, x_r$ in I which is at the same time an unconditioned $M-$sequence in dimension $> s$ and an unconditioned $I-$filter regular sequence with respect to M.*

Proof. We proceed by induction on r. Let $r = 1$. Set

$$C_1 := \big(\operatorname{Ass}_R M\big)_{\geq s+1} \cup \Big(\operatorname{Ass}_R M \setminus \operatorname{Var}(I)\Big).$$

Since $\dim(H_I^0(M)) \leq s$, it follows that $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \big(\operatorname{Ass}_R M\big)_{\geq s+1}$. Therefore, by Prime Avoidance, there exists an element $x_1 \in I$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_1$. It is clear that x_1 is an unconditioned $M-$sequence in dimension $> s$ and an unconditioned $I-$filter regular sequence w.r.t. M.

Let $r > 1$ and assume that the result is true for $r-1$. Then there exists a sequence $x_1, \ldots, x_{r-1}$ in I which is an unconditioned $M-$sequence in dimension $> s$ and an unconditioned $I-$filter regular sequence w.r.t. M. By Lemma 2.4 and by the assumption, for any subset J of $\{1, \ldots, r-1\}$, the sequence $(x_j)_{j \in J}$ can be extended to an $M-$sequence in dimension $> s$ in I of length r. Therefore for any subset J of $\{1, \ldots, r-1\}$, there exists an $(M/\sum_{j \in J} x_j M)-$regular element in dimension $> s$ in I. It follows that $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \Big(\operatorname{Ass}_R \big(M/\sum_{j \in J} x_j M\big)\Big)_{\geq s+1}$ and all subsets J of $\{1, \ldots, r-1\}$. By Prime Avoidance, we can choose an element $x_r \in I$ such that $x_r \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_r$, where

$$C_r := \Big(\bigcup_{J \subseteq \{1, \ldots, r-1\}} \operatorname{Ass}_R(M/\sum_{j \in J} x_j M)\Big)_{\geq s+1} \cup \Big(\bigcup_{J \subseteq \{1, \ldots, r-1\}} \operatorname{Ass}_R(M/\sum_{j \in J} x_j M) \setminus \operatorname{Var}(I)\Big).$$

We first show that $x_1, \ldots, x_r$ is an unconditioned $M-$sequence in dimension $> s$. Let $\sigma \in \mathbb{S}_r$ be a permutation of $1, \ldots, r$. Assume that $x_{\sigma(1)}, \ldots, x_{\sigma(r)}$ is not an $M-$sequence in dimension $> s$. Let $n \in \{1, \ldots, r\}$ be the least integer such that $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ is not an $M-$sequence in dimension $> s$. Then $r = \sigma(i)$ for some $i < n$ by our choice of x_r, and there is some $\mathfrak{p} \in \Big(\operatorname{Ass}_R(M/(x_{\sigma(1)},\ldots,x_{\sigma(n-1)})M)\Big)_{\geq s+1}$ such that $x_{\sigma(n)} \in \mathfrak{p}$. So $x_{\sigma(1)},\ldots,x_{\sigma(n)} \in \mathfrak{p}$ and $\frac{x_{\sigma(1)}}{1},\ldots,\frac{x_{\sigma(n)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_{\sigma(1)}}{1},\ldots,\frac{x_{\sigma(i-1)}}{1},\frac{x_{\sigma(i+1)}}{1},\ldots,\frac{x_{\sigma(n)}}{1},\frac{x_{\sigma(i)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Set $J := \{j \in \mathbb{N} \;:\; j \leq n, j \neq i\}$. As $\dim(R/\mathfrak{p}) > s$, we know that $\Big(\frac{x_{\sigma(j)}}{1}\Big)_{j\in J}$ is a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_r}{1} = \frac{x_{\sigma(i)}}{1}$ is not a regular element w.r.t. $M_{\mathfrak{p}}/\sum_{j\in J} x_{\sigma(j)}M_{\mathfrak{p}}$. So, there exists $\mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\frac{x_r}{1} \in \mathfrak{q}\,R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}\big(M_{\mathfrak{p}}/\sum_{j\in J} x_{\sigma(j)}M_{\mathfrak{p}}\big)$. It follows that $x_r \in \mathfrak{q} \in \Big(\operatorname{Ass}_R(M/\sum_{j\in J} x_{\sigma(j)}M)\Big)_{\geq s+1} \subseteq C_r$. This gives a contradiction.

Finally, we need to prove that $x_1,\ldots,x_r$ is an unconditioned $I-$filter regular sequence w.r.t. M. Assume that $x_{\sigma(1)},\ldots,x_{\sigma(r)}$ is not an $I-$filter regular sequence w.r.t. M for some $\sigma \in \mathbb{S}_r$. Let n be the least integer such that $x_{\sigma(1)},\ldots,x_{\sigma(n)}$ is not an $I-$filter regular sequence w.r.t. M. By our choice of x_r, we have $r = \sigma(i)$ for some $i < n$ and there exists some $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_{\sigma(1)},\ldots,x_{\sigma(n-1)})M) \setminus \operatorname{Var}(I)$ such that $x_{\sigma(n)} \in \mathfrak{p}$. So $x_{\sigma(1)},\ldots,x_{\sigma(n)} \in \mathfrak{p}$ and $\frac{x_{\sigma(1)}}{1},\ldots,\frac{x_{\sigma(n)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_{\sigma(1)}}{1},\ldots,\frac{x_{\sigma(i-1)}}{1},\frac{x_{\sigma(i+1)}}{1},\ldots,\frac{x_{\sigma(n)}}{1},\frac{x_{\sigma(i)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Set $J := \{j \in \mathbb{N} \;:\; j \leq n, j \neq i\}$. As $\mathfrak{p} \notin \operatorname{Var}(I)$, we know that $\Big(\frac{x_{\sigma(j)}}{1}\Big)_{j\in J}$ is a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_r}{1} = \frac{x_{\sigma(i)}}{1}$ is not a regular element w.r.t. $M_{\mathfrak{p}}/\sum_{j\in J} x_{\sigma(j)}M_{\mathfrak{p}}$. So, there exists some $\mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\frac{x_r}{1} \in \mathfrak{q}\,R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}\big(M_{\mathfrak{p}}/\sum_{j\in J} x_{\sigma(j)}M_{\mathfrak{p}}\big)$. It follows that $x_r \in \mathfrak{q} \in \Big(\operatorname{Ass}_R(M/\sum_{j\in J} x_{\sigma(j)}M)\Big)_{\geq s+1} \setminus \operatorname{Var}(I) \subseteq C_r$, a contradiction. □

Proposition 2.6. *Let s be a non-negative integer. Let $x_1,\ldots,x_t$ be an unconditioned $M-$sequence in dimension $> s$. Then*

$$\Big(\bigcup_{n_1,\ldots,n_t\in\mathbb{N}} \operatorname{Ass}_R\big(M/(x_1^{n_1},\ldots,x_t^{n_t})M\big)\Big)_{\geq s} = \Big(\operatorname{Ass}_R(M/(x_1,\ldots,x_t)M))\Big)_{\geq s}.$$

In particular, the set $\Big(\bigcup_{n_1,\ldots,n_t\in\mathbb{N}} \operatorname{Ass}_R\big(M/(x_1^{n_1},\ldots,x_t^{n_t})M\big)\Big)_{\geq s}$ is finite.

Proof. We prove the result by induction on t. Let $t = 1$ and we write $x_1 = x$. We will show by induction on $n_1 = n$ that $\Big(\operatorname{Ass}_R(M/x^nM)\Big)_{\geq s} \subseteq \operatorname{Ass}_R(M/xM)$. The case $n = 1$ is clear.

Let $n>1$ and assume that the result is true for $n-1$. Let $\mathfrak{p}\in\Big(\operatorname{Ass}_R(M/x^nM)\Big)_{\geq s}$. Then $\mathfrak{p}R_{\mathfrak{p}}\in\operatorname{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/x^nM_{\mathfrak{p}})$. If $\dim(R/\mathfrak{p})>s$ then x is a regular element w.r.t. $M_{\mathfrak{p}}$ and therefore $\mathfrak{p}R_{\mathfrak{p}}\in\operatorname{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/xM_{\mathfrak{p}})$. It follows that $\mathfrak{p}\in\operatorname{Ass}_R(M/xM)$. Assume that $\dim(R/\mathfrak{p})=s$. From the exact sequence

$$0\longrightarrow x^{n-1}M/x^nM\longrightarrow M/x^nM\longrightarrow M/x^{n-1}M\longrightarrow 0,$$

we have $\mathfrak{p}\in\operatorname{Ass}_R(M/x^{n-1}M)\cup\operatorname{Ass}_R(x^{n-1}M/x^nM)$. If $\mathfrak{p}\in\operatorname{Ass}_R(M/x^{n-1}M)$ then by induction $\mathfrak{p}\in\operatorname{Ass}_R(M/xM)$. So, assume that $\mathfrak{p}\in\operatorname{Ass}_R(x^{n-1}M/x^nM)$. Consider the exact sequence

$$0\longrightarrow(x^nM:_M x^{n-1})/xM\longrightarrow M/xM\longrightarrow x^{n-1}M/x^nM\longrightarrow 0.$$

Set $K=(x^nM:_M x^{n-1})/xM$. Assume that $\mathfrak{p}\notin\operatorname{Supp}(K)$. Then the above exact sequence shows that $\big(M/xM\big)_{\mathfrak{p}}\cong\big(x^{n-1}M/x^nM\big)_{\mathfrak{p}}$. Since $\mathfrak{p}\in\operatorname{Ass}_R(x^{n-1}M/x^nM)$, it follows that $\mathfrak{p}\in\operatorname{Ass}_R(M/xM)$. So, we assume that $\mathfrak{p}\in\operatorname{Supp}(K)$. Note that for any $\mathfrak{q}\in\operatorname{Supp}(M)$ satisfying $\dim(R/\mathfrak{q})>s$, we have $K_{\mathfrak{q}}=0$ as x is a poor regular element w.r.t. $M_{\mathfrak{q}}$. So $\dim(K)\leq s$. Hence $\mathfrak{p}$ is a minimal element in $\operatorname{Supp}(K)$, and hence $\mathfrak{p}\in\operatorname{Ass}_R(K)\subseteq\operatorname{Ass}_R(M/xM)$. So, the result is true for $t=1$.

Let $t>1$ and assume that the result is true for $t-1$. Let $n_1,\dots,n_t$ be arbitrary positive integers. Since x_t is an $M/(x_1^{n_1},\dots,x_{t-1}^{n_{t-1}})M-$regular element in dimension $>s$, we get by the case $t=1$ that

$$\Big(\operatorname{Ass}_R\big(M/(x_1^{n_1},\dots,x_t^{n_t})M\big)\Big)_{\geq s}\subseteq\operatorname{Ass}_R\big(M/(x_1^{n_1},\dots,x_{t-1}^{n_{t-1}},x_t)M\big).$$

Since $x_1,\dots,x_{t-1}$ is an unconditioned M/x_tM-sequence in dimension $>s$, we have by induction that

$$\begin{aligned}\Big(\operatorname{Ass}_R\big(M/(x_1^{n_1},\dots,x_{t-1}^{n_{t-1}},x_t)M\big)\Big)_{\geq s}&=\Big(\operatorname{Ass}_R\big(M/(x_t,x_1^{n_1},\dots,x_{t-1}^{n_{t-1}})M\big)\Big)_{\geq s}\\&\subseteq\operatorname{Ass}_R\big(M/(x_1,\dots,x_t)M\big).\end{aligned}$$

This proves our claim. □

In general, a permutation of an $M-$sequence in dimension $>s$ is not an $M-$sequence in dimension $>s$. In particular, a permutation of a generalized regular sequence is not necessarily a generalized regular sequence. For example, let $R=k[[x_1,x_2,x_3,x_4,x_5]]$ be the ring of power series in 5 variables over a field k. Let $M=R/(x_1)\cap(x_2,x_3,x_4)$. Then $\dim M=4$ and x_5,x_2 is a generalized regular sequence w.r.t M, but x_2,x_5 is not a generalized regular sequence w.r.t M. This fact shows that Lemma 2.2(i),(ii) of [Nh] is not correct. It follows that the proof of [Nh, 3.1] is not correct, too. The next corollary is an improvement and also a correction of [Nh, 3.1].

Corollary 2.7. *Assume that $x_1,\dots,x_t$ is an unconditioned generalized regular sequence w.r.t. M. Then*

$$\bigcup_{n_1,\dots,n_t\in\mathbb{N}}\operatorname{Ass}_R\big(M/(x_1^{n_1},\dots,x_t^{n_t})M\big)\setminus\operatorname{Max}R\subseteq\operatorname{Ass}_R(M/(x_1,\dots,x_t)M)).$$

In particular, if in addition R is local, then $\bigcup_{n_1,\ldots,n_t\in\mathbb{N}} \mathrm{Ass}_R\left(M/(x_1^{n_1},\ldots,x_t^{n_t})M\right)$ *is a finite set.*

Proof. The proof is immediate by setting $s=1$ in Proposition 2.6. $\square$

3 Finiteness results

Remark 3.1. Let $I \subseteq R$ be an ideal with $\mathrm{depth}(I,M)=t$. Then it is well known that

$$\mathrm{Ass}_R(\mathrm{Ext}^t_R(R/I,M)) = \mathrm{Ass}_R(H^t_I(M)).$$

Lemma 3.2. *Let t be a positive integer. Set $P_t = \bigcup_{i=0}^{t-1} \mathrm{Supp}\left(\mathrm{Ext}^i_R(R/I,M)\right)$. Then*

$$\begin{aligned}&\mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/I^n,M)\right)\cup P_t = \mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/(a_1^{n_1},\ldots,a_k^{n_k}),M)\right)\cup P_t\\ &= \mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/I,M)\right)\cup P_t = \mathrm{Ass}_R(H^t_I(M))\cup P_t\end{aligned}$$

for any system of generators $(a_1,\ldots,a_k)$ of I and all positive integers $n,n_1,\ldots,n_k$.

Proof. Let $\mathfrak{p}\in\mathrm{Supp}(M)$ such that $\mathfrak{p}\notin P_t$. Then for any $i<t$ we have

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p},M_\mathfrak{p}) \cong \left(\mathrm{Ext}^i_R(R/I,M)\right)_\mathfrak{p} = 0.$$

Therefore $\mathrm{depth}(IR_\mathfrak{p},M_\mathfrak{p})\geq t$. Let $n,n_1,\ldots,n_k$ be positive integers. It is clear that

$$\mathrm{depth}(I^nR_\mathfrak{p},M_\mathfrak{p}) = \mathrm{depth}(IR_\mathfrak{p},M_\mathfrak{p}) = \mathrm{depth}((a_1^{n_1},\ldots,a_k^{n_k})R_\mathfrak{p},M_\mathfrak{p}).$$

We first suppose that $\mathrm{depth}(IR_\mathfrak{p},M_\mathfrak{p})>t$. Then the $R_\mathfrak{p}-$modules $\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/I^nR_\mathfrak{p},M_\mathfrak{p})$, $\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/(a_1^{n_1},\ldots,a_k^{n_k})R_\mathfrak{p},M_\mathfrak{p})$, $\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p},M_\mathfrak{p})$, $H^t_{IR_\mathfrak{p}}(M_\mathfrak{p})$ are zero. So, $\mathfrak{p}$ does not belong to any of the four sets $\mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/I^n,M)\right)$, $\mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/(a_1^{n_1},\ldots,a_k^{n_k}),M)\right)$, $\mathrm{Ass}_R\left(\mathrm{Ext}^t_R(R/I,M)\right)$ and $\mathrm{Ass}_R(H^t_I(M))$.

Assume that $\mathrm{depth}(IR_\mathfrak{p},M_\mathfrak{p})=t$. Since $\mathrm{rad}(I)=\mathrm{rad}(I^n)=\mathrm{rad}((a_1^{n_1},\ldots,a_k^{n_k})R)$, we have by Remark 3.1 that

$$\begin{aligned}\mathrm{Ass}_{R_\mathfrak{p}}\left(\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/I^nR_\mathfrak{p},M_\mathfrak{p})\right) &= \mathrm{Ass}_{R_\mathfrak{p}}\left(H^t_{I^nR_\mathfrak{p}}(M_\mathfrak{p})\right) = \mathrm{Ass}_{R_\mathfrak{p}}\left(H^t_{IR_\mathfrak{p}}(M_\mathfrak{p})\right)\\ &= \mathrm{Ass}_{R_\mathfrak{p}}\left(\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p},M_\mathfrak{p})\right),\end{aligned}$$

$$\mathrm{Ass}_{R_\mathfrak{p}}\left(\mathrm{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/(a_1^{n_1},\ldots,a_k^{n_k})R_\mathfrak{p},M_\mathfrak{p})\right) = \mathrm{Ass}_{R_\mathfrak{p}}\left(H^t_{IR_\mathfrak{p}}(M_\mathfrak{p})\right).$$

It follows that

$$\begin{aligned}\mathfrak{p}\in\mathrm{Ass}_R(\mathrm{Ext}^t_R(R/I^n,M) &\Leftrightarrow \mathfrak{p}\in\mathrm{Ass}_R(H^t_I(M)) \Leftrightarrow \mathfrak{p}\in\mathrm{Ass}_R(\mathrm{Ext}^t_R(R/I,M)\\ &\Leftrightarrow \mathfrak{p}\in\mathrm{Ass}_R(\mathrm{Ext}^t_R(R/(a_1^{n_1},\ldots,a_k^{n_k}),M).\end{aligned}$$

Now the result follows immediately. $\square$

The next lemma follows easily by induction on t using Remark 3.1 and Lemma 3.2.

Lemma 3.3. *For any integer $t \geq 0$ we have*

$$\bigcup_{i=0}^{t}\Big(\bigcup_{n\in\mathbb{N}}\operatorname{Supp}\big(\operatorname{Ext}_R^i(R/I^n,M)\big)\Big)=\bigcup_{i=0}^{t}\operatorname{Supp}\big(\operatorname{Ext}_R^i(R/I,M)\big)=\bigcup_{i=0}^{t}\operatorname{Supp}\big(H_I^i(M)\big).$$

It should be mentioned that the equalities $\operatorname{Ass}_R\big(\operatorname{Ext}_R^t(R/I,M)\big)\cup P_t=\operatorname{Ass}_R(H_I^t(M))\cup P_t$ in Lemma 3.2 and $\bigcup_{i=0}^{t}\operatorname{Supp}\big(\operatorname{Ext}_R^i(R/I,M)\big)=\bigcup_{i=0}^{t}\operatorname{Supp}\big(H_I^i(M)\big)$ in Lemma 3.3 have been proved by Cuong-Hoang [CH].

Proof of Theorem 1.1. Let $t \leq r$ be a non-negative integer. Set $P_t=\bigcup_{i=0}^{t-1}\operatorname{Supp}(\operatorname{Ext}_R^i(R/I,M))$. Then $P_t=\bigcup_{i=0}^{t-1}\operatorname{Supp}(H_I^i(M))$ by Lemma 3.3. Therefore, by our assumption, $\dim(R/\mathfrak{p})\leq s$ for all $\mathfrak{p}\in P_t$. Now, let $\mathfrak{p}\in\big(T^t(I,M)\big)_{\geq s}\cup\big(T^t(\underline{a},M)\big)_{\geq s}$. If $\dim(R/\mathfrak{p})>s$ then $\mathfrak{p}\notin P_t$. So, we get by Lemma 3.2 that $\mathfrak{p}\in\operatorname{Ass}_R(\operatorname{Ext}_R^t(R/I,M))$.

Let $\dim(R/\mathfrak{p})=s$. It follows by Lemma 3.2 that $\mathfrak{p}\in\Big(\operatorname{Ass}(\operatorname{Ext}_R^t(R/I,M)\cup P_t\Big)_s$. If $\mathfrak{p}\notin\operatorname{Ass}_R(\operatorname{Ext}_R^t(R/I,M))$ then $\mathfrak{p}\in P_t$, and hence $\mathfrak{p}$ is a minimal prime ideal of $\operatorname{Ext}_R^i(R/I,M)$ for some $i<t$. Therefore $\mathfrak{p}\in\operatorname{Ass}_R(\operatorname{Ext}_R^i(R/I,M))$ for some $i<t$. Thus, $T^t(I,M)$ and $T^t(\underline{a},M)$ are subsets of $\bigcup_{i=0}^{t}\operatorname{Ass}(\operatorname{Ext}_R^i(R/I,M))$. □

Proof of Theorem 1.2. Let $t\leq r$ be a non-negative integer. Let $\mathfrak{p}\in\big(T^t(I,M)\big)_{\geq s}\cup\big(T^t(\underline{a},M)\big)_{\geq s}$. Let $i<t$. Since $\dim(\operatorname{Supp}(H_I^i(M)))\leq s$, any prime ideal in $\Big(\operatorname{Supp}(H_I^i(M))\Big)_{\geq s}$ is minimal in $\operatorname{Supp}(H_I^i(M))$ and hence associated to $H_I^i(M)$. It follows that

$$\Big(\operatorname{Supp}(H_I^i(M))\Big)_{\geq s}=\Big(\operatorname{Ass}_R(H_I^i(M))\Big)_s \text{ for all } i<t.$$

So, we get by Lemma 3.2 and Lemma 3.3 that $\mathfrak{p}\in\Big(\bigcup_{i=0}^{t-1}\operatorname{Ass}_R(H_I^i(M))\Big)_s\cup\Big(\operatorname{Ass}_R(H_I^t(M))\Big)_{\geq s}$. As $x_1,\ldots,x_r\in I$ is an unconditioned $I-$filter regular sequence w.r.t. M, we have by 2.2 that $H_I^i(M)\cong H_I^0(H^i_{(x_1,\ldots,x_i)R}(M))$ for all $i=1,\ldots,t$. Therefore, for each $i=1,\ldots,t$, we have

$$\operatorname{Ass}_R H_I^i(M)\subseteq\operatorname{Ass}_R\big(H^i_{(x_1,\ldots,x_i)R}(M)\big)\subseteq\bigcup_{n\in\mathbb{N}}\operatorname{Ass}_R\big(M/(x_1^n,\ldots,x_i^n)M\big).$$

As $x_1,\ldots,x_r$ is an unconditioned $M-$sequence in dimension $>s$, we get by Proposition 2.6 that

$$\Big(\bigcup_{n\in N}\operatorname{Ass}_R\big(M/(x_1^n,\ldots,x_i^n)M\big)\Big)_s=\Big(\operatorname{Ass}_R(M/(x_1,\ldots,x_i)M)\Big)_s$$

for all $i = 1, \dots, t$, and

$$\Big(\bigcup_{n\in N} \operatorname{Ass}_R \big(M/(x_1^n, \dots, x_t^n)M\big)\Big)_{\geq s+1} = \Big(\operatorname{Ass}_R \big(M/(x_1, \dots, x_t)M\big)\Big)_{\geq s+1}.$$

If $\dim(R/\mathfrak{p}) > s$ then $t = r$ and $\mathfrak{p} \in \operatorname{Ass}_R H_I^t(M)$, hence $\mathfrak{p} \in \Big(\operatorname{Ass}_R(M/(x_1, \dots, x_t)M)\Big)_{\geq s+1}$.
If $\dim(R/\mathfrak{p}) = s$ then

$$\mathfrak{p} \in \bigcup_{i=0}^{t} \Big(\operatorname{Ass}_R(H_I^i(M))\Big)_s = \bigcup_{i=0}^{t} \Big(\operatorname{Ass}_R(M/(x_1, \dots, x_i)M)\Big)_s.$$

Now the result follows immediately. □

Replacing s by 1 in Theorems 1,2 and on use of Lemma 3.2 we get the following result.

Corollary 3.4. *Let $r > 0$ be an integer such that $\dim(\operatorname{Supp}(H_I^i(M))) \leq 1$ for all $i < r$. Let $\underline{a} = (a_1, \dots, a_k)$ be a system of generators of I. Then*

(a) For any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the set

$$\bigcup_{i=0}^{t} \operatorname{Ass}_R(\operatorname{Ext}_R^i(R/I, M)) \cup \Big(\operatorname{Max}(R) \cap \bigcup_{i=0}^{t-1} \operatorname{Supp}(H_I^i(M))\Big).$$

(b) Let $x_1, \dots, x_r \in I$ be an unconditioned generalized regular sequence with respect to M which is an unconditioned $I-$filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the set $\Big(\operatorname{Ass}_R(M/(x_1, \dots, x_t)M)\Big)_{\geq 2} \cup \Big(\bigcup_{i=0}^{t} \operatorname{Ass}_R(M/(x_1, \dots, x_i)M)\Big)_1 \cup \operatorname{Max}(R)$.

Remark 3.5. Let R be local. In Khashyarmanesh [Kh], it is shown that if $H_I^i(M)$ is of finite support for all $i < r$ then the set $\big(T^r(I, M)\big)_{\geq 2} = \{\mathfrak{p} \in T^r(I, M) \ : \ \dim R/\mathfrak{p} > 1\}$ is finite. Corollary 3.4 shows that even $T^r(I, M)$ is a finite set.

Setting $s = 0$ in Theorems 1,2 we get the following result.

Corollary 3.6. *Let $r > 0$ be an integer such that $\dim(\operatorname{Supp}(H_I^i(M))) \leq 0$ for all $i < r$. Let $\underline{a} = (a_1, \dots, a_k)$ be a system of generators of I. Then*

(a) For any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the finite set $\bigcup_{i=0}^{t} \operatorname{Ass}_R(\operatorname{Ext}_R^i(R/I, M))$.

(b) Let $x_1, \dots, x_r \in I$ be an unconditioned filter regular sequence with respect to M which is an unconditioned $I-$filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the finite set $\Big(\operatorname{Ass}_R(M/(x_1, \dots, x_t)M)\Big)_{\geq 1} \cup \bigcup_{i=0}^{t} \operatorname{Ass}_R(M/(x_1, \dots, x_i)M)$.

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