Regularization for the Problem of Finding common Fixed Point of a Finite Family of Nonexpansive Nonself-Mappings in Banach Spaces

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Abstract. In the paper, we study some regularization methods to solve the problem of finding a common fixed point of a finite family of nonexpansive nonself-mappings T_i, $i = 1, 2, \ldots, N$ in an uniformly convex and uniformly smooth Banach space.

Key words: Accretive operators, uniformly smooth and uniformly convex Banach space, sunny nonexpansive retraction, weak sequential continuous mapping, and regularization.

1 Introduction

Let E be a Banach space. We consider the following problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} F(T_i). \tag{1.1}$$

where $T_i : C_i \longrightarrow E$ are the nonexpansive nonself-mappings from a closed convex sunny nonexpansive retract C_i of an uniformly convex and uniformly smooth Banach space E into E $(i = 1, 2, \ldots, N)$.

To solve the problem of finding an element $x^* \in F(T)$, where $F(T)$ is the set of fixed points of nonexpansive nonself-mapping T from a closed convex sunny nonexpansive retract C of a Banach space E into E, S. Matsushita and W. Takahashi [8] considered a iteration method that is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(Q_C T)(x_n), \ n \geq 0. \tag{1.2}$$

where $x, x_0 \in C$ and Q_C is a sunny nonexpansive retraction from E onto C.

In the special case, T is a nonexpansive self-mapping on C, then (1.2) equivalent to

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(x_n), \ x, x_0 \in C, \ n \geq 0. \tag{1.3}$$

which was studied by N. Sioji and W. Takahashi [12]. Note that, the iteration method (1.3) is a extention of Wittmann's result [14] to Banach space.

In addition, the problem of finding a fixed point of a nonexpansive mapping $T : E \longrightarrow E$ is equivalent to the problem of finding a zero of $m-$accretive operator $A = I - T$.

One of the methods to solve the problem $0 \in A(x)$ with A is maximal monotone in Hilbert space H is proximal point algorithm. This algorithm is proposed by Rockafellar [9], starting from any initial guess $x_0 \in H$, this algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J_{c_n}^A(x_n + e_n). \tag{1.4}$$

where $J_r^A = (I + rA)^{-1}$ $\forall r > 0$ is the resolvent of A on the space H. Rockafellar [9] proved the weak convergence of his algorithm (1.4) provided that the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition

$\sum_{n=0}^{\infty} \epsilon_n \| < \infty$. However, Güler's example [7] shows that in infinite dimensional Hilbert space, proximal point algorithm (1.4) has only weak convergence. An example recently of the authors Bauschke, Matoušková and Reich [5] also show that the proximal algorithm only converges weakly but not in norm.

Ryazantseva [10] extended the proximal point algorithm (1.4) for the case that A is a $m-$accretive mapping in a properly Banach space E and proved the weak convergence the sequence of iterations of (1.4) to a solution of the equation $0 \in A(x)$ which is assumed to be unique. Then, to obtain the strong convergence for algorithm (1.4), Ryazantseva [11] combined the proximal algorithm with the regularization, named regularization proximal algorithm, in the form

$$c_n\big(A(x_{n-1}) + \alpha_n x_{n-1}\big) - x_{n-1} = x_n, \ x_0 \in E. \tag{1.5}$$

Under some conditions on c_n and α_n, the strong convergence of $\{x_n\}$ of (1.5) is guaranteed only when the dual mapping j is weak sequential continuous and strong continuous, and the sequence $\{x_n\}$ is bounded.

Attouch and Alvarez [4] considered an extension of the proximal point algorithm (1.4) in the form

$$c_n A(u_{n-1}) + u_{n-1} - u_n = \gamma_n(u_n - u_{n-1}), \ u_0, \ u_1 \in H, \tag{1.6}$$

which is called an inertial proximal point algorithm, where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers. With this algorithm we also only obtained weak convergence of the sequence $\{x_n\}$ to a solution of problem $A(x) \ni 0$ in Hilbert space when the sequences $\{c_n\}$ and $\{\gamma_n\}$ are chosen suitable. Note that this algorithm was proposed by Alvarez in [3] in the context of convex minimization.

The purpose of this paper is to construct an operator version of the Tikhonov regularization method and give a regularization inertial proximal point algorithm to obtain strong convergence of iterative sequences to a solution of the problem (1.1).

2 Preliminaries

Let E be a real Banach space with norm $\|.\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ (resp $x_n \rightharpoonup x$, $x_n \stackrel{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| : \|x\| = \|y\| = 1, \ \|x - y\| = \varepsilon\} \tag{2.1}$$

is called the modulus of convexity of the space E. The function $\delta_E(\varepsilon)$ defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$, $\forall \varepsilon \in (0, 2]$.

The function

$$\rho_E(\tau) = \sup\{2^{-1}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = \tau\}. \tag{2.2}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive.

A mapping J from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.4}$$

is called the normalized duality mapping of E. In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$, and if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J is single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, $J(x_n) \stackrel{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

An operator $A : D(A) \subset E \longrightarrow 2^E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0, \ \forall u \in A(x), \ v \in A(y) \tag{2.5}$$

An operator $A : E \longrightarrow 2^E$ is called $m-$accretive if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$, where I denote the identity of E. If A is a $m-$accretive operator in Banach space E with E has a weakly sequentially continuous duallity mapping J, then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightharpoonup x$ and $A(x_n) \ni y_n \longrightarrow f$, then $A(x) = f$ [2].

A mapping $T : C \longrightarrow E$ is called nonexpansive mapping on a closed convex subset C of a Banach space E if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C \tag{2.6}$$

If $T : C \longrightarrow E$ is a nonexpansive mapping then $I - T$ is accretive operator. In the casse the subset C coincides E then $I - T$ is $m-$accretive operator [6].

A mapping Q of C into C is said to be a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of E and let Q be a mapping of C into D. Then Q is said to be sunny if each point on the ray $\{Qx + t(x - Qx) : t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t > 0$ and $x \in C$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D.

A closed convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Proposition 2.1. [1] *Let C be a nonempty closed convex subset of a smooth Banach E. A mapping $Q_C : E \longrightarrow C$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_C x, J(\xi - Q_C x)\rangle \leq 0, \ \forall x \in E, \ \forall \xi \in C \tag{2.7}$$

3 Main results

We need the following lemmas in the proof of our results.

Lemma 3.1. [3] *Let $\{a_n\}, \{b_n\}, \{\sigma_n\}$ be the sequences of positive numbers satisfying the conditions*

i) $a_{n+1} \leq (1 - b_n)a_n + \sigma_n, \; b_n < 1;$

ii) $\sum_{n=0}^{\infty} b_n = +\infty, \; \lim_{n\to\infty} \sigma_n / b_n = 0.$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 3.2. *Let C be a closed convex subset of a strictly convex Banach space E and let $T : C \longrightarrow E$ be a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $F(T) \neq \emptyset$, then $F(T) = F(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C.*

Lemma 3.3. [1] *Let E be an uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping $T : D(T) \longrightarrow E$, then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \geq L^{-1} R^2 \delta_E\left(\frac{\|Ax - Ay\|}{4R}\right), \tag{3.1}$$

where $\|x\| \leq R$, $\|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Theorem 3.4. *Suppose that E is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a closed convex sunny nonexpansive retract of E and let $T_i : C_i \longrightarrow E$, $i = 1, 2, \ldots, N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$.*

i) *For each $\alpha_n > 0$ the equation*

$$\sum_{i=1}^{N} A_i(x_n) + \alpha_n x_n = 0 \tag{3.2}$$

has unique solution x_n, where $A_i = I - Q_{C_i} T_i Q_{C_i}$, $i = 1, 2, \ldots, N$ and $Q_{C_i} : E \longrightarrow C_i$ is a sunny nonexpansive retraction form E onto C_i, $i = 1, 2, \ldots, N$;

ii) *If, in addition, $\alpha_n \longrightarrow 0$ then $x_n \longrightarrow Q_S\theta$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S and θ is origin of space E.*

Moreover, we have the following estimate

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} R_0, \tag{3.3}$$

where $R_0 = 2\|Q_S\theta\|$.

Proof. i) First, it is clear that T_iQ_C is a nonexpansive mapping on E and $F(T_i) = F(T_iQ_C)$, $i = 1, 2, \dots, N$. Hence, $Q_CT_iQ_C$ are also nonexpansive mappings for all $i = 1, 2, \dots, N$. By Lemma 3.2, we have $F(T_i) = F(Q_CT_iQ_C)$, $i = 1, 2, \dots, N$. Thus the problem (1.1) is equivalent to the problem of finding a common zero of operators A_i, $i = 1, 2, \dots, N$. Since the operator $\sum_{i=1}^N A_i$ is Lipschitz continuous and accretive on E, it is m-accretive [6]. Therefore the equation (3.2) has unique solution x_n.
ii) For each $x^* \in S$, we have

$$\sum_{i=1}^{N} \langle A_i(x_n), j(x_n - x^*)\rangle + \alpha_n\langle x_n, j(x_n - x^*)\rangle = 0. \tag{3.4}$$

By the accretiveness of $\sum_{i=1}^N A_i$, we obtain

$$\langle x_n, j(x_n - x^*)\rangle \le 0. \tag{3.5}$$

The obtained inequatlity yields the estimates

$$\|x_n - x^*\|^2 \le \langle x^*, j(x_n - x^*)\rangle \le \|x^*\| \times \|x_n - x^*\| \tag{3.6}$$

Hence, $\|x_n\| \le 2\|x^*\|$, i.e., the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subset \{x_n\}$ and an element $\overline{x} \in E$ such that $x_{n_k} \rightharpoonup \overline{x}$ as $k \to \infty$.

We will show $\overline{x} \in S$. Indeed, for each $i \in \{1, 2, \dots, N\}$, $x^* \in S$ and $R > 0$ satisfy $R \ge \max\{\sup\|x_n\|, \|x^*\|\}$ we have

$$\begin{aligned}\delta_E\left(\frac{\|A_i(x_n)\|}{4R}\right) &\le \frac{L}{R^2}\langle A_i(x_n), j(x_n - x^*)\rangle \le \frac{L}{R^2}\langle\sum_{k=1}^{N} A_k(x_n), j(x_n - x^*)\rangle \\ &\le \frac{L\alpha_n}{R^2}\|x_n\|.\|x_n - x^*\| \le \frac{L\alpha_n}{R^2}2\|x^*\|^2 \longrightarrow 0, \ n \to \infty.\end{aligned}$$

By the continuity of the function $\delta_E(.)$ and the uniformly convexity of Banach space E, we obtain $A_i(x_n) \longrightarrow 0$, $n \longrightarrow \infty$. Hence $A_i(\overline{x}) = 0$ from the demiclosedness of A_i. Since $i \in \{1, 2, \dots, N\}$ is the arbitrary element, so $\overline{x} \in S$.
In inequality (3.6) replacing x_n by x_{n_k} and x^* by $\overline{x}$, using the weak continuity of j we obtain $x_{n_k} \longrightarrow \overline{x}$. From inequality (3.5) we get

$$\langle \overline{x}, j(\overline{x} - x^*)\rangle \le 0, \ \forall x^* \in S \tag{3.7}$$

Now, we show that the inequality (3.7) has unique solution. Suppose that $\overline{x}_1 \in S$ is also its solution. Then

$$\langle \overline{x}_1 - y, j(\overline{x}_1 - x^*)\rangle \le 0, \ \forall x^* \in S \tag{3.8}$$

In inequalities (3.7) and (3.8) replacing x^* by $\overline{x}_1$ and $\overline{x}$, respectively, we obtain

$$\begin{aligned}&\langle \overline{x} - y, j(\overline{x} - \overline{x}_1)\rangle \le 0, \\ &\langle y - \overline{x}_1, j(\overline{x} - \overline{x}_1)\rangle \le 0.\end{aligned}$$

Their combination gives $\|\overline{x} - \overline{x}_1\|^2 \le 0$, thus $\overline{x} = \overline{x}_1 = Q_S\theta$ and the sequence $\{x_n\}$ converges weakly to $\overline{x} = Q_S\theta$, because $Q_S\theta$ satisfies the inequality (3.7).

Finally, we will prove the inequality (3.3). In equation (3.2), replacing n by $n+1$, we obtain

$$\sum_{i=1}^{N} A_i(x_{n+1}) + \alpha_{n+1}x_{n+1} = 0. \tag{3.9}$$

From equations (3.9) and (3.2) and by the accretiveness of the operator $\sum_{i=1}^{N} A_i$, we get

$$\langle \alpha_{n+1}x_{n+1} - \alpha_n x_n, j(x_{n+1} - x_n)\rangle \le 0. \tag{3.10}$$

Therefore,

$$\begin{aligned}\alpha_n\|x_{n+1} - x_n\|^2 &\le (\alpha_{n+1} - \alpha_n)\langle -x_{n+1}, j(x_{n+1} - x_n)\rangle\\ &\le |\alpha_{n+1} - \alpha_n|.\|x_{n+1}\|.\|x_{n+1} - x_n\|\\ &\le 2\|Q_S\theta\|.|\alpha_{n+1} - \alpha_n|.\|x_{n+1} - x_n\|.\end{aligned}$$

Hence,

$$\|x_{n+1} - x_n\| \le \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}R_0, \ \forall n \ge 0,$$

where $R_0 = 2\|Q_S\theta\|$. □

Next, we give a regularization inertial proximal point algorithm in the form

$$c_n\left(\sum_{i=1}^{N} A_i(u_{n+1}) + \alpha_n u_{n+1}\right) + u_{n+1} = u_n + \gamma_n(u_n - u_{n-1}), \ u_0, \ u_1 \in E \tag{3.11}$$

to solve the problem (1.1).

Theorem 3.5. *Suppose that E is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a closed convex sunny nonexpansive retract of E and let $T_i : C_i \longrightarrow E$, $i = 1, 2, \dots, N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the sequences $\{c_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n$;

ii) $\alpha_n > 0, \alpha_n \longrightarrow 0, \ \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} \longrightarrow 0, \ \sum_{n=0}^{\infty} \alpha_n = +\infty$;

iii) $\gamma_n \ge 0, \ \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by equation (3.11) converges strongly to $Q_S\theta$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, we show that the equation (3.11) has unique solution u_{n+1}. Indeed, since the operator $\sum_{i=1}^{N} A_i$ is Lipschitz continuous and accretive on E, it is $m-$accretive [6]. Therefore the equation (3.11) has unique solution u_{n+1}.

Now, we rewrite the equations (3.2) and (3.11) in their equivalent forms

$$d_n \sum_{i=1}^{N} A_i(x_n) + x_n = \beta_n x_n, \tag{3.12}$$

$$d_n \sum_{i=1}^{N} A_i(u_{n-1}) + u_{n-1} = \beta_n(u_n - \gamma_n(u_n - u_{n-1})), \tag{3.13}$$

where $\beta_n = \dfrac{1}{1 + c_n\alpha_n}$ and $d_n = c_n\beta_n$.

From equations (3.13) and (3.12) and by virtue of the property of $\sum_{i=1}^{N} A_i$ we have

$$\|u_{n-1} - x_n\| \le \beta_n\|u_n - x_n\| + \beta_n\gamma_n\|u_n - u_{n-1}\|.$$

Therefore,

$$\begin{aligned}\|u_{n-1} - x_{n-1}\| &\le \|u_{n-1} - x_n\| + \|x_{n-1} - x_n\| \\ &\le \beta_n\|u_n - x_n\| + \beta_n\gamma_n\|u_n - u_{n-1}\| + \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n}R_0.\end{aligned} \tag{3.14}$$

or equivalent to

$$\|u_{n+1} - x_{n+1}\| \le (1 - b_n)\|u_n - x_n\| + \sigma_n, \; b_n = \frac{c_n\alpha_n}{1 + c_n\alpha_n} \tag{3.15}$$

where $\sigma_n = \beta_n\gamma_n\|u_n - u_{n-1}\| + \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}R_0$.

Under the assumption we have

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + \left(\frac{1}{c_n} + \alpha_n\right)\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2}R_0 \\ &\le \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + \left(\frac{1}{c_0} + \alpha_n\right)\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2}R_0 \longrightarrow 0.\end{aligned}$$

Furthermore, by $\sum_{n=0}^{\infty}\alpha_n = +\infty$ hence $\sum_{n=0}^{\infty} b_n = +\infty$.

By Lemma 3.1, $\|u_n - x_n\| \to 0$. Since $x_n \to Q_S y$ as $n \longrightarrow \infty$, $u_n \to Q_S y$ as $n \to \infty$. □

Summary

Hiệu chỉnh bài toán tìm điểm bất động chung của một họ hữu hạn các ánh xạ không giãn trong không gian Banach

Abstract. Trong bài báo này, chúng tôi nghiên cứu một số phương pháp hiệu chỉnh cho bài toán tìm điểm bất động chung của một họ hữu hạn các ánh xạ không giãn T_i, $i = 1, 2, \ldots, N$ trong không gian Banach lồi đều và trơn đều.

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