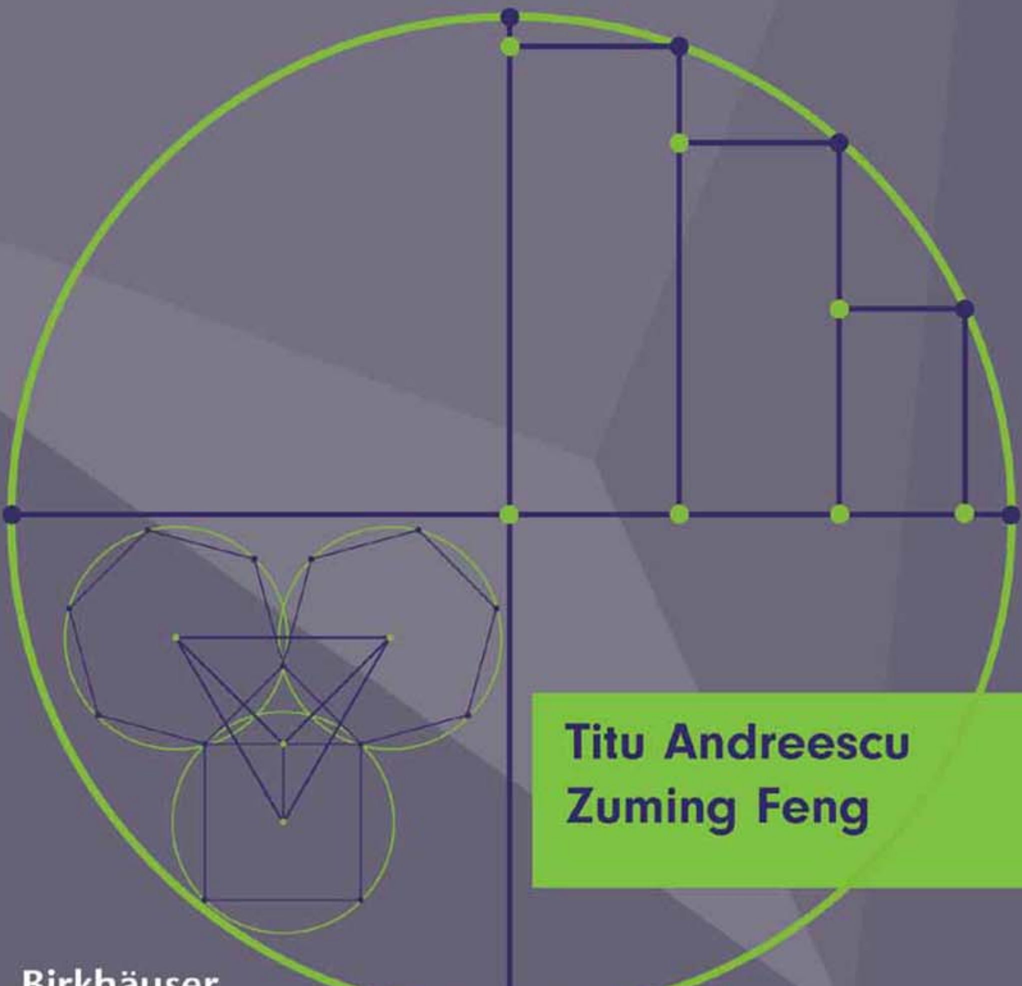
103 TRIGONOMETRY PROBLEMS

From the Training of the USA IMO Team



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About the Authors

Titu Andreescu received his BA, MS, and PhD from the West University of Timisoara, Romania. The topic of his doctoral dissertation was "Research on Diophantine Analysis and Applications". Titu served as director of the MAA American Mathematics Competitions (1998-2003), coach of the USA International Mathematical Olympiad Team (IMO) for 10 years (1993-2002), director of the Mathematical Olympiad Summer Program (1995-2002) and leader of the USA IMO Team (1995-2002). In 2002 Titu was elected member of the IMO Advisory Board, the governing body of the international competition. Titu received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994 and a "Certificate of Appreciation" from the president of the MAA in 1995 for his outstanding service as coach of the Mathematical Olympiad Summer Program in preparing the US team for its perfect performance in Hong Kong at the 1994 International Mathematical Olympiad.

Zuming Feng graduated with a PhD from Johns Hopkins University with emphasis on Algebraic Number Theory and Elliptic Curves. He teaches at Phillips Exeter Academy. He also served as a coach of the USA IMO team (1997-2003), the deputy leader of the USA IMO Team (2000-2002), and an assistant director of the USA Mathematical Olympiad Summer Program (1999-2002). He is a member of the USA Mathematical Olympiad Committee since 1999, and has been the leader of the USA IMO team and the academic director of the USA Mathematical Olympiad Summer Program since 2003. He received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1996 and 2002.

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Preface

This book contains 103 highly selected problems used in the training and testing of the U.S. International Mathematical Olympiad (IMO) team. It is not a collection of very difficult, impenetrable questions. Instead, the book gradually builds students' trigonometric skills and techniques. The first chapter provides a comprehensive introduction to trigonometric functions, their relations and functional properties, and their applications in the Euclidean plane and solid geometry. This chapter can serve as a textbook for a course in trigonometry. This work aims to broaden students' view of mathematics and better prepare them for possible participation in various mathematical competitions. It provides in-depth enrichment in important areas of trigonometry by reorganizing and enhancing problem-solving tactics and strategies. The book further stimulates interest for the future study of mathematics.

In the United States of America, the selection process leading to participation in the International Mathematical Olympiad (IMO) consists of a series of national contests called the American Mathematics Contest 10 (AMC 10), the American Mathematics Contest 12 (AMC 12), the American Invitational Mathematics Examination (AIME), and the United States of America Mathematical Olympiad (USAMO). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence. The Mathematical Olympiad Summer Program (MOSP) is a four-week intensive training program for approximately 50 very promising students who have risen to the top in the American Mathematics Competitions. The six students representing the United States of America in the IMO are selected on the basis of their USAMO scores and further testing that takes place during MOSP.

Throughout MOSP, full days of classes and extensive problem sets give students thorough preparation in several important areas of mathematics. These topics include combinatorial arguments and identities, generating functions, graph theory, recursive relations, sums and products, probability, number theory, polynomials, functional equations, complex numbers in geometry, algorithmic proofs, combinatorial and advanced geometry, functional equations, and classical inequalities.

Olympiad-style exams consist of several challenging essay problems. Correct solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved with elementary high school mathematics techniques, cleverly applied.

Here is some advice for students who attempt the problems that follow.

- Take your time! Very few contestants can solve all the given problems.

- Try to make connections between problems. An important theme of this work is that all important techniques and ideas featured in the book appear more than once!

- Olympiad problems don't "crack" immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backwards from the desired result is helpful.

- Even if you can solve a problem, do read the solutions. They may contain some ideas that did not occur in your solutions, and they may discuss strategic and tactical approaches that can be used elsewhere. The solutions are also models of elegant presentation that you should emulate, but they often obscure the tortuous process of investigation, false starts, inspiration, and attention to detail that led to them. When you read the solutions, try to reconstruct the thinking that went into them. Ask yourself, "What were the key ideas? How can I apply these ideas further?"

- Go back to the original problem later, and see whether you can solve it in a different way. Many of the problems have multiple solutions, but not all are outlined here.

- Meaningful problem-solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use the books on the reading list.

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Many of the ideas of the first chapter are inspired by the Math 2 and Math 3 teaching materials from the Phillips Exeter Academy. We give our deepest appreciation to the authors of the materials, especially to Richard Parris and Szczesny "Jerzy" Kaminski.

Many problems are either inspired by or adapted from mathematical contests in different countries and from the following journals:

- *High-School Mathematics*, China
- *Revista Matematică Timişoara*, Romania

We did our best to cite all the original sources of the problems in the solution section. We express our deepest appreciation to the original proposers of the problems.