DETERMINATION OF A TIME-DEPENDENT TERM IN THE RIGHT HAND SIDE OF LINEAR PARABOLIC EQUATIONS

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ABSTRACT

We propose a variational method for determining a time-dependent term in the right hand side of parabolic equations from integral observations. It is proved that the functional to be minimized is Fréchet differentiable and a formula for its gradient is derived via an adjoint problem. The problem is discretized by the finite difference methods and then he conjugate gradient method in coupling with Tikhonov regularization is applied for numerically solving it. Numerical results are presented showing that our technique is efficient.

Keywords: *Inverse problems, ill-posed problems, integral observations, finite difference method, conjugate gradient method.*

1 Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$. Denote by $\partial\Omega$ the boundary of Ω, $Q := \Omega\times(0,T]$, and $S := \partial\Omega \times (0,T]$. Consider the following problem

$$\begin{cases} u_t - \Delta u &= f(t)\varphi(x,t) + \psi(x,t), (x,t) \in Q,\\ u(x,0) &= u_0(x), x \in \Omega,\\ \dfrac{\partial u}{\partial \nu} &= g(x,t), (x,t) \in S. \end{cases} \tag{1.1}$$

Here, ν is the outer normal to $\partial\Omega$. And $\varphi \in L^\infty(Q)$, $\psi(x,t) \in L^2(Q)$, $u_0(x) \in L^2(\Omega)$, $g(x,t) \in L^2(S)$. Furthermore, it is assumed that $\varphi > \underline{\varphi} > 0$, with $\underline{\varphi}$ being a given constant.

To introduce the concept of weak solution, we use the standard Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H^{1,0}(Q)$ and $H^{1,1}(Q)$ [3, 5, 6]. Further, for a Banach space B, we define $L^2(0,T;B) = \{u : u(t) \in B \text{ a.e. } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;B)} < \infty\}$, with the norm

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B dt.$$

In the sequel, we shall use the space $W(0,T)$ defined as $W(0,T) = \{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2(0,T;(H^1(\Omega))')\}$, equipped with the norm $\|u\|^2_{W(0,T)} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(0,T;(H^1(\Omega))')}$.

Definition 1. The function $u \in W(0,T)$ is said to be a weak solution of (1.1) if for all $\eta \in L^2(0,T;H^1(\Omega))$ satisfying $\eta(\cdot,T) = 0$, the following identity holds

$$\int_0^T \langle u_t, \eta\rangle_{(H^1(\Omega))',H^1(\Omega)} dt + \iint_Q \nabla u \nabla \eta dxdt + \iint_S \alpha g(x,t)\eta(x,t)dsdt = \iint_Q (f(t)\varphi(x,t) + \psi(x,t))\,\eta(x,t)dxdt, \tag{1.2}$$

and $u|_{t=0} = u_0$. It is proved in [5] that there exists a unique solution in $W(0,T)$ of the problem (1.1). Furthermore, there is a positive constant $c_d > 0$ independent of f, φ, ψ, g and u_0 such that

$$\|u\|_{W(0,T)} \le c_d(\|f\varphi\|_{L^2(Q)} + \|\psi\|_{L^2(Q)} + \|g\|_{L^2(S)} + \|u_0\|_{L^2(\Omega)}).$$

In this paper, we will consider the inverse problem of determining the time-dependent term

$f(t)$ from an integral observation of the solution u. Namely, we try to reconstruct $f(t)$ from the observation

$$lu(x,t) := \int_\Omega \omega(x)u(x,t)dx = h(t),\ t \in (0,T), \tag{1.3}$$

where $\omega(x)$ is a weight function. We suppose that $\omega \in L^\infty(\Omega)$, nonnegative almost everywhere in Ω and $\int_\Omega \omega(x)dx > 0$. The observation data h is supposed to be in $L^2(0,T)$. We will reformulate the inverse problem in our setting to a variational problem and prove that the functional to be minimized is Fréchet differentiable and derive a formula for it. To solve the variational problem numerically we discretize it by the finite difference method, and then use the conjugate gradient method (CG) in coupling with Tikhonov regularization to stabilize the solution and then test the algorithm on computer.

2 Main result

2.1 Variational formulation

From now on, to emphasize the dependence of the solution u on the unknown function f, we write $u(f)$ or $u(x,t;f)$ instead of u. Following the least-squares approach [1, 2, 4], we estimate the unknown function $f(t)$ by minimizing the objective functional

$$J_0(f) = \frac{1}{2} \parallel lu(f) - h \parallel^2_{L_2(0,T)} \tag{2.1}$$

over $L^2(0,T)$.

To stabilize this variational problem, we minimize the Tikhonov functional

$$J_\gamma(f) = \frac{1}{2}\|lu(f) - h\|^2_{L_2(0,T)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(0,T)} \tag{2.2}$$

with γ being a regularization parameter which has to be properly chosen and f^* an estimation of f which is supposed in $L^2(0,T)$. We have a result:

Theorem 1 *The functional J_γ is Fréchet differentiable and its gradient $\nabla J_\gamma(f)$ at f has the form*

$$\nabla J_\gamma(f) = \int_\Omega p(x,t)\varphi(x,t)dx + \gamma(f(t) - f^*(t)), \tag{2.3}$$

where $p(x,t)$ satisfies the adjoint problem

$$\begin{cases} -p_t - \Delta p = \omega(x)\left(lu(x,t;f) - h(t)\right) \text{ in } Q, \\ p(x,T) = 0, x \in \Omega, \\ \dfrac{\partial p}{\partial \nu} = 0, (x,t) \in S. \end{cases} \tag{2.4}$$

Proof. For an infinitesimally small variation δf of f, we have

$$J_0(f+\delta f) - J_0(f) = \frac{1}{2}\|lu(f+\delta f) - h\|^2_{L_2(0,T)}$$
$$- \frac{1}{2}\|lu(f) - h\|^2_{L_2(0,T)}$$
$$= \langle l\delta u(f), lu(f) - h\rangle + \frac{1}{2}\|l\delta u(f)\|^2_{L_2(0,T)},$$

where $\delta u(f)$ is the solution to the problem

$$\begin{cases} \delta u_t - \Delta \delta u &= \delta f(t)\varphi(x,t), (x,t) \in Q, \\ \delta u(x,0) &= 0, x \in \Omega, \\ \dfrac{\partial \delta u}{\partial \nu} &= 0, (x,t) \in S. \end{cases} \tag{2.5}$$

We have the estimate, $\|l\delta u(f)\|^2_{L_2(0,T)} = o(\|\delta f\|_{L^2(0,T)})$ as $\|\delta f\|_{L^2(0,T)} \to 0$.

We have

$$J_0(f+\delta f) - J_0(f)$$
$$= \langle l\delta u, lu - h\rangle + o(\|\delta f\|_{L^2(0,T)})$$
$$= \int_0^T \Big(\int_\Omega \omega\delta u dx\Big)\big(lu - h\big)dt + o(\|\delta f\|_{L^2(0,T)})$$
$$= \int_0^T \Big(\int_\Omega \omega\delta u(lu-h)dx\Big)dt + o(\|\delta f\|_{L^2(0,T)})$$
$$= \int_0^T \int_\Omega \omega\delta u(lu-h)dxdt + o(\|\delta f\|_{L^2(0,T)}).$$

Using Green's formula (see, i.e. [5, Theorem 3.18]) for (2.5) and (2.4), we have

$$\int_0^T \int_\Omega \omega\delta u(lu-h)dxdt = \int_0^T \int_\Omega \delta f\varphi p dxdt.$$

Hence,

$$
\begin{aligned}
&J_0(f+\delta f)-J_0(f)\\
&=\int_0^T\int_\Omega \delta f\varphi p dx dt+o(\|\delta f\|_{L^2(0,T)})\\
&=\Big\langle \int_\Omega \varphi(x,t)p(x,t)dx,\delta f\Big\rangle+o(\|\delta f\|_{L^2(0,T)}).
\end{aligned}
$$

Consequently, J_0 is Fréchet differentiable and its gradient has the form

$$\nabla J_0(f)=\int_\Omega \varphi(x,t)p(x,t)dx.$$

From this equality, we immediately arrive at (2.3). The proof is complete.

The direct and inverse problem are solved using the finite difference method. Now we introduce the conjugate gradient method for solving the variational problem.

2.2 Conjugate gradient method

Step 1: 1.1. Given an initial approximation $f^0\in\mathbb{R}^n$.

1.2. Calculate $U^0(f^0)$ is the solution of the direct problem

$$
\begin{cases}
U_t^0-\Delta U^0=f^0(t)\varphi(x,t)+\psi(x,t)\text{in } Q,\\
U^0(x,0)=u_0(x), x\in\Omega,\\
\dfrac{\partial U^0}{\partial\nu}=g(x,t),(x,t)\in S.
\end{cases}
$$

1.3. Calculate the residual $\tilde r_0 = lU^0(f^0)-h=\int_\Omega \omega(x)U^0(x,t;f^0)dx-h$ with $\omega(x)$ is the weight and $h=\int_\Omega\omega(x)u_{ex}(x,t;f^0)dx$, for u_{ex} is the exact solution of (1.1).

1.4. Calculate $r_0=-\nabla J_\gamma(f^0)$ given in (2.3) by solving the adjoint

$$
\begin{cases}
-p_t^0-\Delta p^0=\omega(x)\big(lu(x,t;f^0)-h(t)\big)\text{ in } Q,\\
p^0(x,T)=0, x\in\Omega,\\
\dfrac{\partial p^0}{\partial\nu}=0,(x,t)\in S.
\end{cases}
$$

1.5. Set $d_0=r_0$.

Step 2: For $n=0,1,2,...$

2.1. Calculate $Ad_n = lU^n(d_n) = \int_\Omega\omega(x)U^n(x,t;d_n)dx$, here $U^n(x,t;d_n)$ is the solution of the direct problem.

2.2. Calculate $\alpha_n=\dfrac{||\,r_k\,||^2_{L^2(0,T)}}{||Ad_n||^2_{L^2(0,T)}+\lambda\,||d_n||^2_{L^2(0,T)}}.$

2.3. Update $f_{n+1}=f_n+\alpha_n d_n$.

2.4. Calculate the residual $\tilde r_{n+1}=\tilde r_n+\alpha_n Ad_n$.

2.5. Calculate the gradient r_{n+1} given in (2.3) by solving the adjoint

$$
\begin{cases}
-p_t^{n+1}-\Delta p^{n+1}=\omega(x)\left(lu(x,t;f^n)-h(t)\right)\\
p^{n+1}(x,T)=0, x\in\Omega,\\
\dfrac{\partial p^{n+1}}{\partial\nu}=0,(x,t)\in S.
\end{cases}
$$

2.6. Calculate $\beta_n=\dfrac{||r_{n+1}||^2_{L^2(0,T)}}{||r_n||^2_{L^2(0,T)}}.$

2.7. Update $d_{n+1}=r_n+\beta_n d_n$.

2.3 Numerical simulation

In this we present some numerical examples showing that our algorithm is efficient. Let $\Omega=(0,1)$. We reconstruct the function f from the system for $x\in(0,1), t\in(0,1)$

$$
\begin{cases}
u_t-u_{xx}=f(t)\varphi(x,t)+g(x,t),\\
u(x,0)=u_0(x),\\
-u_x(0,t)=g_1(t), u_x(1,t)=g_2(t).
\end{cases}\qquad(2.6)
$$

Let $T=1$, we test our algorithm for reconstructing the functions in three cases: the first $f(t)$ is smooth, the second $f(t)$ not differentiable at $t=0.5$ and the last one is discontinuous.

Example 1: $f(t)=\sin(\pi t)$.

Example 2: $f(t)=\begin{cases}2t & \text{if } 0\le t\le 0.5\\ 2(1-t) & \text{if } 0.5\le t\le 1.\end{cases}$

Example 3: $f(t)=\begin{cases}1 & \text{if } 0.25\le t\le 0.75\\ 0 & \text{otherwise }.\end{cases}$

We take $u(x,t) = \sin(\pi t)\cos(x-t)$, $u_0(x) = 0$, $g_1(t) = \sin(\pi t)\sin(-t)$, $g_2(t) = sin(\pi t)\sin(1-t)$, $\varphi(x,t) = (x^2+1)(t^2+1)$ and then put one of the above functions f into the system to get $g(x,t)$. In the observation lu (2.6) we take the following weight functions

$$\omega(x) = x^2 + 1 \qquad (2.7)$$

The numerical results for these tests are presented in Figures 1-3. From these results we see that the numerical results in the one dimensional cases are very good, although the noise level is 10%.

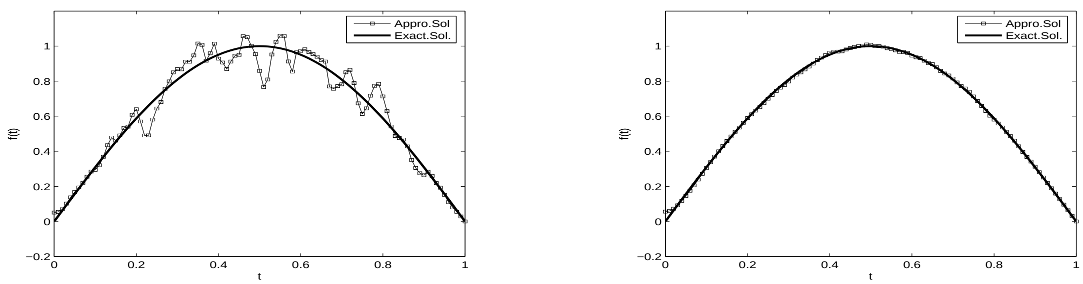

Figure 1. Example 1: The exact solution in comparison with the numerical solution with noise level = 0.1 (left) and noise level = 0.01 (right). The weight function ω is given by (2.7).

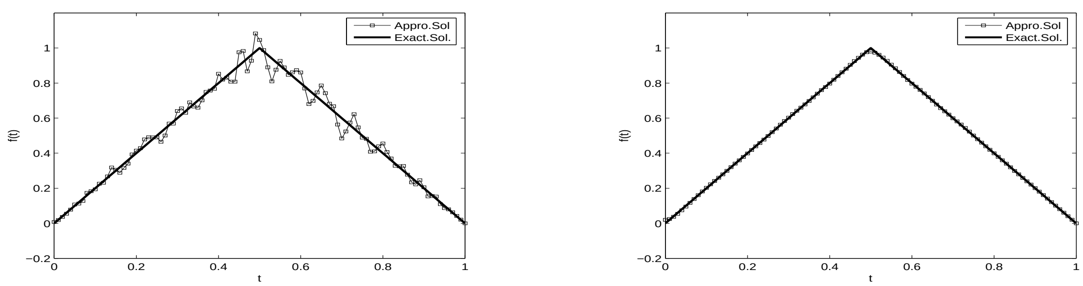

Figure 2. Example 2: The exact solution in comparison with the numerical solution with noise level = 0.1 (left) and noise level = 0.01 (right). The weight function ω is given by (2.7).

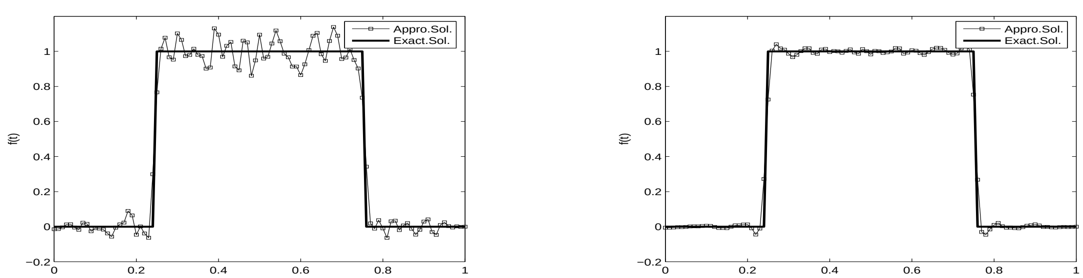

Figure 3. Example 3: The exact solution in comparison with the numerical solution with noise level = 0.1 (left) and noise level = 0.01 (right). The weight function ω is given by (2.7).

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TÓM TẮT

XÁC ĐỊNH MỘT THÀNH PHẦN PHỤ THUỘC VÀO THỜI GIAN TRONG VẾ PHẢI CỦA PHƯƠNG TRÌNH PARABOLIC

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Tóm tắt

Chúng tôi đề xuất phương pháp biến phân cho bài toán xác định một thành phần phụ thuộc thời gian trong phương trình parabolic từ các quan sát tích phân. Chúng tôi chứng minh phiếm hàm cần tối thiểu hóa khả vi Fréchet và đưa ra công thức xác định nó qua bài toán liên hợp. Bài toán được rời rạc bằng phương pháp sai phân và được giải số bằng phương pháp gradient liên hợp kết hợp với phương pháp chỉnh Tikhonov. Một số ví dụ bằng số thực hiện trên máy tính đã thể hiện tính hiệu quả của phương pháp.

Từ khóa : *Bài toán ngược, bài toán đặt không chỉnh, quan sát tích phân, phương pháp sai phân, phương pháp gradient liên hợp.*

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