MODEL REDUCTION IN SCHUR BASIS WITH POLE RETENTION AND H∞-NORM ERROR BOUND

Dao Huy Du[1*], Ha Binh Minh[2]
[1]College of Technology – TNU,
[2]Hanoi University of Science and Technology, VN

SUMMARY

We propose a new algorithm to obtain a reduced model with pole retention. The main idea is that instead of transforming A into diagonal matrix as in modal truncation technique, we transform A into upper-triangle matrix. The H∞-norm error bound of this algorithm is given. The choice of pole retention will be discussed to get reduced model having minimal H∞-norm error bound.
Key words: *linear time-invariant systems, modal truncation, pole retention, H∞-norm error bound, triangle realization, triangle truncation.*

INTRODUCTION

Modal approximation is simple and effective technique in model reduction. This technique retains a part of the poles of original system. The reduced model therefore retains some physical interpretations of the original one, such as some vibration modes. Modal approximation technique also provides an error bound formula, which is useful to give the first estimation of how many state or pole need to be discasted.

Modal approximation techniqueis based on selecting the poles which are important for model reduction's purposes. There are two ways to select these poles. The first one can be classified as "top-down" methods, in which we search every poles and then select the important ones. Modal truncation method, which is discussed in Section 2, belongs to this class. The second one can be classified as "bottom-up" methods, in which we search pole one-by-one and then compare the new pole we found to the set of poles found before. If the new pole is better than others then we select, otherwise we discaste. Some numerical methods developed recently in [7,8] belong to this class.

The aim of this paper is to improve modal truncation method. The idea of truncation' can be divided into two steps: first,

transforming original system to equivalent system by an onsigular transformation in the statespace, and second, deleting some rows and columns to get a reduced system. In modal truncation method, matrix A is transformed into diagonal form. Our improvement idea isasfollows. In stead of transforming A into diagonal form, we transform A into upper-triangle form by Schur decomposition. The advantage of Schur decomposition is that it is relizable and reduce computational cost.

The structure of this paper is as follows. Modal truncation method will be reviewed in Section 2. A new realization, which is called triangle realization, will be presented in Section 3. In Section 4, we discuss algorithm to get reduced-order model based on new realization and errorbound. In Section 5, numerical example will be presented. Finally, conclusions will be given in Section 6.

REVIEW OF MODAL TRUNCATION METHOD

Consider a linear time-invariant system represented by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \tag{1}$$

where:

$$x \in R_n, u \in R_p, y \in R_q, A \in R_{nxn}, B \in R_{nxp}, C \in R_{qxn}, D \in R_{qxp}$$

Here, we assume that system (1) takes values in C instead of in R due to simplicity. The transfer function of system (1) is given by

* Tel: 0912 347222, Email: daohuydu@tnut.du.vn

Mục tiêu của bài toán giảm bậc mô hình là tìm ra mô hình mô tả bởi hệ các phương trình:

$$G(s) := C(sI - A)^{-1}B$$

Assume that the transfer function G(s) of system is asymptotically stable and in minimal realization. It is always possible to perform a state-space transformation

$$x_{\mathrm{mod}}(t) = T^{-1}x(t),$$

$$\left(A_{\mathrm{mod}}, B_{\mathrm{mod}}, C_{\mathrm{mod}}\right) = \left(T^{-1}AT, T^{-1}B, CT\right),$$

such that Amod is in diagonal form

$$A_{\mathrm{mod}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, B_{\mathrm{mod}} = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix},$$

$$C_{\mathrm{mod}} = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} \qquad (2)$$

where, for simplicity, we assume that each of the eigenvalues λi of A has a simple Jordan structure. The realization above is called diagonal realization. The nice thing of this realization is that transfer function G(s) can be factorized as the sum of simple transfer function.

$$G(s) = G_1(s) + G_2(s) + \cdots + G_n(s), \qquad (3)$$

where $G_i(s) := C_i(s - \lambda_i)^{-1}B_i$, $i = 1...n$ are simple transfer functions. The H_∞ norm of G(s) can be estimated via those of $G_i(s)$, i = 1...n, as follows.

$$\|G(s)\|_{\mathrm{H}_\infty} \leq \sum_{i=1}^{n}\|G_i(s)\|_{\mathrm{H}_\infty} = \sum_{i=1}^{n}\frac{\|C_iB_i\|_2}{\mathrm{Re}\,\lambda_i} \qquad (4)$$

where $\|C_iB_i\|_2 = \sigma_{max}(C_iB_i)$

The right hand side of (4) tell us that each pole contribute a term $\dfrac{\|C_iB_i\|_2}{|\mathrm{Re}\,\lambda_i|}$ to the H_∞-norm of G(s). In many control purposes, the poles contribute much maybe more important than the pole Zpoles.

Definition 1 *[7] For given G(s) in diagonal realization (2), the pole λi of G(s) is called dominant if its corresponding term*

$$\hat{R}_i = \frac{\|C_iB_i\|_2}{|\mathrm{Re}\,\lambda_i|} \text{ is relatively large compared to}$$

others $\hat{R}_j$, $j \neq i$. *The term* $\hat{R}_i$ *is called the dominance index of pole λi.*

Now we turn to modal truncation technique. The objective of modal truncation is to divide the set of eigenvalues of A, i.e. the poles of G(s), into two sets: one to be discarded and the other to be kept in the reduced-order system. Suppose that we want to keep r eigenvalues of A in the set $\Lambda_r = \{\lambda_{i1},...,\lambda_{ir}\}$, then the reduced-order system obtained via the modal truncation is

$$G_{red}(s) = \sum_{\lambda_i \in \Lambda_r} G_i(s) = \sum_{\lambda_i \in \Lambda_r} C_i(s - \lambda_i)^{-1}B_i$$

The H_∞-norm error bound in modal truncation technique is given by (see, e.g. [8]).

$$\|G(s) - G_{red}(s)\|_{\mathrm{H}_\infty} \leq \sum_{\lambda_i \notin \Lambda_r} \frac{\|C_iB_i\|_2}{|\mathrm{Re}\,\lambda_i|} = \sum_{\lambda_i \notin \Lambda_r} \hat{R}_i$$

$$(5)$$

In order to get small H∞-norm error bound, we should discard the poles having their dominance index $\hat{R}_i$ small, or keep the poles having their dominance index large. We summarize modal truncation technique as follows.

Algorithm 1 (Modal truncation technique)
Assume that the linear time-invariant system (1) is asymptotically stable and in a minimal representation (A,B,C).

Step 1: Use transformation T to obtain $(A_{\mathrm{mod}}, B_{\mathrm{mod}}, C_{\mathrm{mod}})$ in diagonal realization (2).

Step 2: For each pole λi, i = 1...n, compute its dominance index $\hat{R}_i$. Then, arrange them in decreasing order

$$\hat{R}_{i_1} \geq \cdots \geq \hat{R}_{i_r} \geq \hat{R}_{i_{r+1}} \geq \cdots \geq \hat{R}_{i_n}.$$

Step 3: Select r dominant poles to be retained in reduced order system: Λr = {λi1,...,λir}. The reduced-order system is given as follows

$$G_{red}(s) = \sum_{\lambda_i \in \Lambda_r} G_i(s) = \sum_{\lambda_i \in \Lambda_r} C_i(s - \lambda_i)^{-1}B_i.$$

TRIANGLE REALIZATION

The main purpose of this section is to obtain a realization in which A has triangle form. However, this realization should have properties similar to diagonal realization in some senses: (i) transfer function G(s) can be

factorized similar to (3); (ii) dominant pole is defined similar to Definition 1; and (iii) the error bound formula is similar to (5). We propose the following algorithm to obtain this realization.

Algorithm 2 *(Triangle realization)*
Assume that the linear time-invariant system (1) is asymptotically stable and in a minimal representation (A,B,C).

Input: $(A, B, C) \in C^{n \times n} \times C^{n \times m} \times C^{p \times n.}$.

Step 1: Compute observability Gramian Q from Lyapunov equation $A * Q + QA + C * C = 0$.

Step 2: Compute Cholesky factorization $Q = R^* R$.

Step 3: Compute Schur decomposition of RAR^{-1}: $RAR^{-1} = U\Delta U^*$, where U is unitary matrix and Δ is upper triangle matrix.

Step 4: Compute nonsingular transformation $T = R^{-1}U$.

Step 5: Compute $\left(\tilde{A}, \tilde{B}, \tilde{C}\right) = \left(T^{-1}AT, T^{-1}B, CT\right)$.

Output: An equivalent system with realization $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$.

Definition 2 The output realization $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ in Algorithm 2 is said to be triangle realization.

The triangle realization $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ has many nice properties, which will be investigated in this section. The following lemma will explain why $\tilde{A}$ is triangle matrix. It also investigates the controllability and observability Gramians of this realization. For the role of controllability and observability Gramians in model reduction, especially in balanced truncation, we refer the reader to [2].

Lemma 3 The triangle realization $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ in Algorithm 2 has the following properties:

(a) The matrix $\tilde{A}$ is upper triangle matrix.

(b) The observability Gramian $\tilde{Q}$ is identity matrix.

(c) The controllability Gramian $\tilde{P}$ can be factorized as $\tilde{P} = W^* \Sigma^2 W$, where W is unitary matrix, and

$\Sigma = diag\left(\sigma_1^2, ..., \sigma_n^2\right)$ is diagonal matrix whose diagonal entries are the square of the Hankel singular values of the system.

PROOF. First, we verify that $\tilde{A}$ is in upper triangle form. In deed,

$\tilde{A} = T^{-1}AT = (U^* R)A\left(R^{-1}U\right) = U^* U\Delta U^* U = \Delta$,

which is upper triangle matrix.

Next, we prove that the new observability Gramian $\tilde{Q}$ is identity matrix. In deed,

$$\tilde{Q} = T^* QT = (U^* R^{-*})Q(R^{-1}U) =$$
$$(U^* R^{-*})(R^* R)(R^{-1}U) = U^* U = I.$$

The new controllability Gramian $\tilde{P}$ is

$$\tilde{P} = T^{-1}PT^{-*} = (U^* R)P(R^* U^{-*}) = U^* RPR^* U$$

where P is the controllability Gramian in realization (A,B,C). Take the singular value decomposition of RPR^* as $RPR^* = V^* \Sigma^2 V$, where V is unitary matrix. Now $\tilde{P}$ can be rewrite as follows

$$\tilde{P} = U^* V^* \Sigma^2 V \, U = W^* \Sigma^2 W ,$$

where $W := V \, U$ is unitary matrix.

Remark 4: We will show how to transform triangle realization to balanced realization. Assume that $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ is in triangle realization, which have controllability Gramians $\tilde{P} = W^* \Sigma^2 W$ and observability Gramians $\tilde{Q} = I$, as shown in Lemma 3. We use the transformation $S := W^* \Sigma^{1/2}$ to obtain an equivalent balanced realization $\left(S^{-1}AS, S^{-1}B, CS\right)$. Indeed, the controllability and observability Gramians of realization $\left(S^{-1}AS, S^{-1}B, CS\right)$ respectively are:

$$P_{bal} = S^{-1}\tilde{P}S^{-*}$$
$$= (\Sigma^{-1/2}W^{-*})(W^* \Sigma^2 W)(W^* \Sigma^{-1/2}) = \Sigma;$$

$$Q_{bal} = S^* \tilde{Q}S = \Sigma^{1/2}WW^* \Sigma^{1/2} = \Sigma,$$

which implies that $\left(S^{-1}AS, S^{-1}B, CS\right)$ is balanced realization.

Factorize G(s) as the sum of two transfer functions

In this subsection, we investigate that how G(s) can be factorized as the sum of two simpler transfer functions using triangle realization $\left(\tilde{A}, \tilde{B}, \tilde{C} \right)$. By partitioning $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{P}, \tilde{Q}$ into 2×2 block matrices with appropriate dimensions, with the note that in general case $\tilde{P}$ is not in diagonal form, we can rewrite two Lyapunov equations $\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^* + \tilde{B}\tilde{B}^* = 0$ and $\tilde{A}^*\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{C}^*\tilde{C} = 0$ as follows.

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{11}^* & \tilde{P}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{11}^* & \tilde{P}_{22} \end{bmatrix} \begin{bmatrix} \tilde{A}_{11}^* & 0 \\ \tilde{A}_{12}^* & \tilde{A}_{22}^* \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{B}_1\tilde{B}_1^* & \tilde{B}_1\tilde{B}_2^* \\ \tilde{B}_2\tilde{B}_1^* & \tilde{B}_2\tilde{B}_2^* \end{bmatrix} = 0$$

$$\begin{bmatrix} \tilde{A}_{11}^* & 0 \\ \tilde{A}_{12}^* & \tilde{A}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{C}_1^*\tilde{C}_1 & \tilde{C}_1^*\tilde{C}_2 \\ \tilde{C}_2^*\tilde{C}_1 & \tilde{C}_2^*\tilde{C}_2 \end{bmatrix} = 0$$

These are equivalent to the following equations

$$\tilde{A}_{11}\tilde{P}_{11} + \tilde{A}_{12}\tilde{P}_{12}^* + \tilde{P}_{11}\tilde{A}_{11}^* + \tilde{P}_{12}\tilde{A}_{12}^* + \tilde{B}_1\tilde{B}_1^* = 0, \qquad (6)$$

$$\tilde{A}_{11}\tilde{P}_{12} + \tilde{A}_{12}\tilde{P}_{22} + \tilde{P}_{12}\tilde{A}_{22}^* + \tilde{B}_1\tilde{B}_2^* = 0, \qquad (7)$$

$$\tilde{A}_{22}\tilde{P}_{22} + \tilde{P}_{22}\tilde{A}_{22}^* + \tilde{B}_2\tilde{B}_2^* = 0, \qquad (8)$$

$$\tilde{A}_{11}^* + \tilde{A}_{11} + \tilde{C}_1^*\tilde{C}_1 = 0, \qquad (9)$$

$$\tilde{A}_{12} + \tilde{C}_1^*\tilde{C}_2 = 0, \qquad (10)$$

$$\tilde{A}_{22}^* + \tilde{A}_{22} + \tilde{C}_2^*\tilde{C}_2 = 0, \qquad (11)$$

Let $H(s) := \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$ and $K(s) := \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2$ respectively be the trasfer functions of two subsystems. The following lemma will show how transfer function G(s) can be factorized as the sum of H(s) and K(s).

Lemma 5 With notations given in this subsection we get that

$$G(s) = H(s) + V(s)K(s),$$

where $V(s) := I - \tilde{C}_1\left(sI - \tilde{A}_{11}\right)^{-1}\tilde{C}_1^*$.

Moreover, V(s) has property that $V(-s)*V(s) = I$.

Factorize G(s) as the sum of n transfer functions

Assume that in triangle realization $\left(\tilde{A}, \tilde{B}, \tilde{C} \right)$ has the following form

$$\tilde{A} = \begin{bmatrix} \lambda_1 & * & * \\ & \ddots & * \\ 0 & & \lambda_n \end{bmatrix}, \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_n \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_n \end{bmatrix} \qquad (12)$$

Set $G_i(s) := \tilde{C}_i(s - \lambda_i)^{-1}\tilde{B}_i, i = 1...n$ and $V_i(s) := I - \tilde{C}_i(s - \lambda_i)^{-1}\tilde{C}_1^*, i = 1...n$. Then we get the following lemma.

Lemma 6 With notations given in this subsection we get that

$$G(s) = G_1(s) + V_1(s)G_2(s) + V_1(s)V_2(s)G_3(s)$$
$$+ \cdots + V_1(s)V_2(s) \cdots V_{n-1}(s)G_n(s)$$
$$(13)$$

Moreover, $V_i(s)$ has property that

$$V_i(-s)^* V_i(s) = I, i = 1...n-1.$$

Remark 7 *In the right hand side of (13) the first term $G_1(s)$ has only one pole λ1, the second term $V_1(s)G_2(s)$ has two poles $λ_1, λ_2$, and so on, the last term $V_1(s)...V_{n-1}(s)G_n(s)$ has n poles $λ_1,...,λ_n$. Compare to (3), each term in the right hand side of (3) has only one pole $λ_i$.*

Dominant pole in triangle realization

In this subsection we will show how to determine dominant pole from triangle realization. Definition 1 is not used in this case since it only aplly for diagonal realization. Therefore we need a new definiton of dominant pole in triangle realization.

To make the definition of dominant pole, we use the view point that dominant poles contribute much to the H∞ norm of G(s) (see also the discussion before Definition 1). Therefore we need to estimate the H∞-norm of G(s) before giving the definition.

Lemma 8 *Assume that G(s) is in triangle realization (12) and notations are given as in Lemma 6. Then*

$$\|G(s)\|_{H_\infty} \leq \|G_1(s)\|_{H_\infty} + \|G_2(s)\|_{H_\infty} + \cdots + \|G_n(s)\|_{H_\infty}$$

$$= \frac{\|\tilde{C}_1\tilde{B}_1\|_2}{|\operatorname{Re}\lambda_1|} + \frac{\|\tilde{C}_2\tilde{B}_2\|_2}{|\operatorname{Re}\lambda_2|} + \cdots + \frac{\|\tilde{C}_n\tilde{B}_n\|_2}{|\operatorname{Re}\lambda_n|}$$

(14)

The right hand side in formula (14) tell us that

each pole contribute a term $\dfrac{\|\tilde{C}_i\tilde{B}_i\|_2}{|\operatorname{Re}\lambda_i|}$ to the H_∞-

norm of G(s). This leads to the following definition of dominant pole in triangle realization.

Definition 9 For given G(s) in triangle realization (2), thepole λ_i of G(s) is called dominant if its corresponding term

$\tilde{R}_i := \dfrac{\|\tilde{C}_i\tilde{B}_i\|_2}{|\operatorname{Re}\lambda_i|}$ is relatively large compared to

others $\tilde{R}_j$, $j \neq i$. The term $\tilde{R}_i$ is called the dominance index of pole λ_i.

Remark 10 *The dominant poles in Definition 9 and Definition 1 depend on different realizations. However, they share the same meaning that they are the poles having the most contribution to the H_∞-norm of G(s). It is still an open question that the dominant poles in two definitions are the same or not. This question is beyond the scope of this paper and should be considered independently in other research. See [1] for more dicussion of dominant poles.*

TRIANGLE TRUNCATION TECHNIQUE

Assume that G(s) is in triangle realization (12), which is as follows.

$$\tilde{A} = \begin{bmatrix} \lambda_1 & * & * \\ & \ddots & * \\ 0 & & \lambda_n \end{bmatrix}, \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_n \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_n \end{bmatrix}.$$

Recall from Lemma 6 that

$$G(s) = G_1(s) + V_1(s)G_2(s) + V_1(s)V_2(s)G_3(s)$$

$$+ \cdots + V_1(s)V_2(s)\cdots V_{n-1}(s)G_n(s)$$

The terms in the right hand side of above formula increase in McMillan degree, that means the first term *G₁(s)* has McMillan degree 1, the second term *V₁(s)G₂(s)* has McMillan degree 2, and so on, the last term

V₁(s)...Vₙ₋₁(s)Gₙ(s) has McMillan degree n. If we want to make reduced system with McMillan degree r, r < n, we should take the first *r* terms as follows.

$$G_{red}(s) = G_1(s) + V_1(s)G_2(s) + \cdots$$

$$+ \cdots + V_1(s)\cdots V_{r-1}(s)G_r(s)$$

(17)

Is that the reduced system *G*red(*s*) has McMillan degree *r*? The answer will be given in the following theorem.

Theorem 11 *The reduced system Gred(s) in (17) has the following properties:*

(a) Gred(s) has triangle realization $\left(\tilde{A}_{red}, \tilde{B}_{red}, \tilde{C}_{red}\right)$, *where* $\left(\tilde{A}_{red}, \tilde{B}_{red}, \tilde{C}_{red}\right)$ *are received by truncating from* $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ *the last n−r rows and columns, i.e.,*

$$\tilde{A}_{red} = \begin{bmatrix} \lambda_1 & * & * \\ & \ddots & * \\ 0 & & \lambda_r \end{bmatrix}, \tilde{B}_{red} = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_r \end{bmatrix},$$

$$\tilde{C}_{red} = \begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_r \end{bmatrix}.$$

(b) The H_∞-norm of Gred(s) can be estimated as follows.

$$\|G_{red}(s)\|_{H_\infty} \leq \|G_1(s)\|_{H_\infty} + \|G_2(s)\|_{H_\infty} + \cdots + \|G_r(s)\|_{H_\infty}$$

$$= \frac{\|\tilde{C}_1\tilde{B}_1\|_2}{|\operatorname{Re}\lambda_1|} + \frac{\|\tilde{C}_2\tilde{B}_2\|_2}{|\operatorname{Re}\lambda_2|} + \cdots + \frac{\|\tilde{C}_r\tilde{B}_r\|_2}{|\operatorname{Re}\lambda_r|}$$

(19)

(c) The H_∞-norm of G(s)−Gred(s) can be estimated as follows.

$$\|G(s) - G_{red}(s)\|_{H_\infty} \leq \|G_{r+1}(s)\|_{H_\infty} + \cdots + \|G_n(s)\|_{H_\infty}$$

$$= \frac{\|\tilde{C}_{r+1}\tilde{B}_{r+1}\|_2}{|\operatorname{Re}\lambda_{r+1}|} + \cdots + \frac{\|\tilde{C}_n\tilde{B}_n\|_2}{|\operatorname{Re}\lambda_n|}$$

(20)

By error bound formula (20), we should choose the reduced system *G*red(*s*) having dominant poles, i.e., poles having their dominance index large. To do that we need to reorder the position of poles on the diagonal of $\tilde{A}$, as shown in the following algorithm

Algorithm 3 *(Triangle truncation technique)*
Input: Triangle realization $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$ of *G(s)*, which is the output of Algorithm 2.

Step 1: For each pole λ_i, i = 1...n, compute its dominance index $\tilde{R}_i := \dfrac{\left\| \tilde{C}_i \tilde{B}_i \right\|_2}{\left| \text{Re} \, \lambda_i \right|}$. Then, arrange them in de-creasing order

$$\tilde{R}_{i_1} \geq \cdots \geq \tilde{R}_{i_r} \geq \tilde{R}_{i_{r+1}} \geq \cdots \geq \tilde{R}_{i_n}.$$

Step 2: *Select r dominant poles to be retained in reduced order system: $\Lambda_r = \{\lambda_{i1},...,\lambda_{ir}\}$.*

Step 3: *Reorder the eigenvalue in the diagonal of $\tilde{A}$ by unitary matrix V such that the r chosen poles $\{\lambda_{i1},...,\lambda_{ir}\}$ lie on the top of diagonal and the rest lie on the bottom of diagonal.*

$$V^* \tilde{A} V = \begin{bmatrix} \lambda_{i_1} & * & * & * & * & * \\ & \ddots & * & * & * & * \\ & & \lambda_{i_r} & * & * & * \\ & & & \lambda_{i_{r+1}} & * & * \\ & & & & \ddots & * \\ & & & & & \lambda_{i_n} \end{bmatrix}$$

(**Comment:** *Algorithms for reordering eigenvalues in Schur decomposition are referred to [3,5]. It can be done, for example, by MATLAB function ordschur.*)

Step 4: *Compute $\left(\hat{A}, \hat{B}, \hat{C} \right) = (V^* \tilde{A} V, V^* \tilde{B}, \tilde{C} V)$*

Step 5: *Truncate the n − r last rows and columns of $\left(\hat{A}, \hat{B}, \hat{C} \right)$ to get*

$$\hat{A}_{red} = \begin{bmatrix} \lambda_{i_1} & * & * \\ & \ddots & * \\ 0 & & \lambda_{i_r} \end{bmatrix}, \hat{B}_{red} = \begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_r \end{bmatrix},$$

$$\hat{C}_{red} = \begin{bmatrix} \hat{C}_1 & \cdots & \hat{C}_r \end{bmatrix}. \qquad (21)$$

Output: *A reduced system $G_{red}(s)$ with realization $\left(\hat{A}_{red}, \hat{B}_{red}, \hat{C}_{red} \right)$.*

NUMERICAL EXAMPLE

We test the triangle truncation technique for the following system with an eighth-order model of a flexible structure [10]. Other tests for this system can be found in [4, Sec. 9.6].

$$G(s) = \sum_{i=1}^{4} k_i \frac{\omega_i^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}$$

where

i	ω_i	ζ_i	k_i
1	0.56806689746895	0.00096819582773	0.01651378989774
2	3.94093897440699	0.00100229920475	0.00257034576009
3	10.58229653714164	0.00100167293203	0.00002188016252
4	16.19234386986640	0.01000472824082	0.00027927762861

The poles and their dominant indecies $\tilde{R}_i := \dfrac{\left\| \tilde{C}_i \tilde{B}_i \right\|_2}{\left| \text{Re} \, \lambda_i \right|}$, $i =1,...,8$ in Definition 9 are given as follows.

i	λ_i	$\tilde{R}_i$	Ordering by $\tilde{R}_i$
1	−0.1620+16.1915i	0.0140	#5
2	−0.1620−16.1915i	0.0140	#6
3	−0.0040+3.9409i	1.2822	#3
4	−0.0040−3.9409i	1.2822	#4
5	−0.0006+0.5681i	8.5281	#1
6	−0.0006−0.5681i	8.5281	#2
7	−0.0106+10.5823i	0.0109	#7
8	−0.0106−10.5823i	0.0109	#8

If four poles are eliminated, we should eliminate poles λ_1, λ_2, λ_7, λ_8 to obtain a reduced model as

$$\hat{A}_{red} = \begin{bmatrix} -0.0040 & -3.9449 & -0.0029 & -0.0029 \\ 3.9370 & -0.0040 & -0.0029 & -0.0029 \\ 0 & 0 & -0.0006 & -0.5686 \\ 0 & 0 & 0.5675 & -0.0006 \end{bmatrix} \hat{B}_{red} = \begin{bmatrix} -0.0570 \\ 0.0570 \\ -0.1415 \\ 0.1413 \end{bmatrix},$$

$$\hat{C}_{red} = \begin{bmatrix} -0.0889 & -0.0889 & -0.033 & -0.033 \end{bmatrix}$$

Figure 1 shows the full-order model, the fourth-order reduced model and the error of two models. Two high frequency modes are deleted, corresponding to λ_1, λ_2 and $\lambda_7, \lambda_8.$
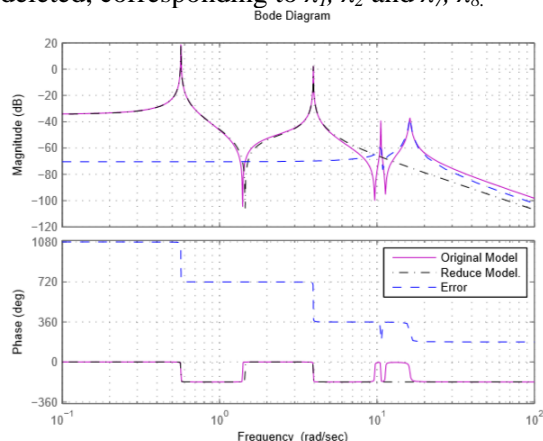
Fig. 1. *Model reduction by triangle truncation technique: full-order model (solid), reduced model (dash-dot) and their error (dashed).*

CONCLUSIONS

Different from modal truncation or balanced truncation techniques, where eigenvalue decomposition or singular value decomposition are used, the algorithms presented in this paper use only Schur decomposition. Although the algorithms may not success for every system, this is the first attemp for triangularizing matrix A. The main contribution of this paper is focused on theoritical aspect of this triangularization approach.

It should be noted that there are still some open questions to be answered. The first question involving about dominant poles are described in Remark 10. The second question

is about contructing similar algorithms for system (1) whose *(A,B,C)* are real matrices. In this case, the real Schur form or Hessenberg form will be used. These questions will be answers in our other researchs.

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TÓM TẮT
GIẢM BẬC MÔ HÌNH THEO SCHUR
VÀ ĐÁNH GIÁ SAI SỐ THEO CHUẨN H$_\infty$

Đào Huy Du[1*], Hà Bình Minh[2]
[1]Trường Đại học Kỹ thuật Công nghiệp – ĐH Thái nguyên
[2]Trường Đại học Bách Khoa Hà Nội

Chúng tôi đề xuất một thuật toán mới cho mô hình giảm bậc mà vẫn bảo toàn điểm cực. Ý tưởng là đưa ma trận trạng thái về dạng tam giác như trong phương pháp model truncation, chuyển đổi ma trận trạng thái A có dạng tam giác trên. Trong bài báo này, tác giả sử dụng chuẩn để đánh giá sai số. Việc lựa chọn bảo toàn điểm cực bị ràng buộc dựa trên sai số tối thiểu.
Từ khóa: *bảo toàn điểm cực, chuẩn sai số H$_\infty$, ma trận tam giác*

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[*] *Tel: 0912 347222, Email: daohuydu@tnut.du.vn*