Nonlinear Functional Analysis and Applications
Vol. 20, No. 2 (2015), pp. 187-197

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

A GENERALIZED QUASI-RESIDUAL PRINCIPLE IN REGULARIZATION FOR A SOLUTION OF A FINITE SYSTEM OF ILL-POSED EQUATIONS IN BANACH SPACES

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Abstract. In this paper we present a generalized quasi-residual principle to select a value for regularization parameter in the Browder–Tikhonov regularization method, for finding a solution of a system of ill-posed equations involving potential, hemicontinuous and monotone mappings on Banach spaces. An estimate of convergence rates for regularized solution is also established.

1. Introduction

Let E be a real reflexive Banach space and E^* be its dual space, which both are assumed to be strictly convex. For the sake of simplicity, norms of E and E^* are denoted by the symbol $\|.\|$ and $\langle x^*, x\rangle$ denotes the value of the linear and continuous functional $x^* \in E^*$ at the point $x \in E$. When $\{x_n\}$ is a sequence in E, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence.

[0]Received October 18, 2014. Revised December 14, 2014.
[0]2010 Mathematics Subject Classification: 47J05, 47H09, 49J30.
[0]Keywords: Monotone, strictly monotone, λ-inverse strongly-monotone mapping, reflexive Banach space, Fréchet differentiable, Browder–Tikhonov regularization.

In addition, we assume that E possesses the ES-property: weak convergence and convergence in norms for any sequence in E follow its strong convergence.

Consider the problem of finding a solution for a system of the following equations

$$A_i(x) = f_i, \quad f_i \in E^*, \ i = 0, 1, \ldots, N, \tag{1.1}$$

where N is a fixed positive integer and A_i is a potential, hemicontinuous and monotone mapping on E, i.e., $\mathcal{D}(A_i) \equiv E$ for $i = 0, 1, \ldots, N$, and $\mathcal{D}(A)$ denotes the domain of A. Recall that a mapping A of domain $\mathcal{D}(A) \subseteq E$ into E^* is called λ-inverse-strongly monotone, iff

$$\langle A(x) - A(y), x - y\rangle \geq \lambda \|A(x) - A(y)\|^2, \quad \forall\, x, y \in \mathcal{D}(A),$$

where λ is a positive constant.

The examples of inverse-strongly monotone operators in the Banach space setting can see in [3].

A is called monotone, iff A satisfies the following condition

$$\langle A(x) - A(y), x - y\rangle \geq 0, \quad \forall\, x, y \in \mathcal{D}(A);$$

strictly monotone at a point $y \in \mathcal{D}(A)$, iff the equality in the last inequality follows $x = y$; and potential, iff $A(x) = \varphi'(x)$, the Gâteaux derivative of a convex functional $\varphi(x)$.

Denote by S_i the set of solutions for ith equation in (1.1). Throughout this paper, we assume that $S := \bigcap_{i=0}^{N} S_i \neq \emptyset$. We are specially interested in the situation where the data f_i is not exactly known, i.e., we have only the approximations $f_i^\delta \in E^*$, satisfying

$$\|f_i - f_i^\delta\| \leq \delta, \quad \delta \to 0, \tag{1.2}$$

for $i = 0, 1, \ldots, N$.

It is well-known in [1] that each equation in (1.1), in general, is ill-posed, by this we mean that the solutions do not depend continuously on the data f_i. Consequently, the system of equations (1.1), in general, is ill-posed. Many practical inverse problems are naturally formulated in such a way and some methods are studied for solving (1.1) (see, [4]–[7]). In 2006, to solve (1.1) in the case that $f_i = \theta$-the null element in E^*, and A_i is a potential, hemicontinuous and monotone mapping on E, in [8], Buong presented the regularization method of Browder–Tikhonov type:

$$\sum_{i=0}^{N} \alpha^{\mu_j} A_i^h(x) + \alpha U(x) = \theta, \qquad \mu_0 = 0 < \mu_i < \mu_{i+1} < 1, \quad i = 1, 2, \dots, N-1, \tag{1.3}$$

where A_i^h is a hemicontinuous and monotone approximation for A_i, U is the normalized duallity mapping of E, i.e., $U : E \to 2^{E^*}$, that satisfies the condition

$$\langle U(x), x\rangle = \|x\|\|U(x)\| \quad \text{and} \quad \|U(x)\| = \|x\|,$$

for all $x \in E$, and α is a regularization parameter, whose value $\alpha = \alpha(h)$ is selected by the equation $\tilde{\rho}(\alpha) = \alpha^{-q}h^p$ with $\tilde{\rho}(\alpha) = \alpha(a_0 + \|x_\alpha^h\|)$ and a_0, q, p are some fixed positive constants. Further, in [9] and [10], method (1.3) was modified for the case, when A_0 is a Lipschitz continuous and monotone mapping and the other A_i is a λ_i-inverse-strongly monotone mapping in Hilbert spaces.

For the stated problem, as in [8], we consider the following equation

$$\sum_{i=0}^{N} \alpha^{\mu_i}(A_i(x) - f_i^\delta) + \alpha U(x - x^+) = \theta, \qquad \mu_0 = 0 < \mu_i < \mu_{i+1} < 1, \quad i = 1, 2, \dots, N-1, \tag{1.4}$$

where the initial point $x^+ \notin S$. Formulating a procedure to numerically implement (1.4) we can use an explicit method that are similar (27) and (28) in [2].

Clearly, the mapping $A(.) := \sum_{i=0}^{N} \alpha^{\mu_i}(A_i(.) - f_i^\delta) + \alpha U$, for each fixed $\alpha > 0$, is hemicontinuous and monotone with $\mathcal{D}(A) = E$. Hence, A is maximal monotone (see [1], Theorem 1.4.6). So, equation (1.4) possesses a unique solution x_α^δ, for each $\alpha > 0$. By the similar argument, as in [8], we have that if α, $\delta/\alpha \to 0$ then x_α^δ converges strongly to $x_0 \in S$, satisfying

$$\|x_0 - x^+\| = \min_{z \in S} \|z - x^+\|. \tag{1.5}$$

In this paper, we consider a choice $\overline{\alpha} = \alpha(\delta)$ by using the principle

$$\rho(\alpha) := \alpha\|x_\alpha^\delta - x^+\| = \alpha^{-q}\delta^p, \tag{1.6}$$

where p, q are some positive constants and estimate convergence rates for $x_{\alpha(\delta)}^\delta$ under the following conditions:

$$\|A_0(y) - f_0 - A_0'(x_0)^*(y - x_0)\| \leq \tau\|A_0(y) - f_0\|, \tag{1.7}$$

for y in some neighbourhood of $x_0 \in S$, where $A_0'(x)$ denotes the derivative of A_0 at $x \in E$, $A_0'(x)^*$ is the adjoint of $A_0'(x)$, τ is some positive constant, and

$$\langle U(x) - U(y), x - y\rangle \geq m_U\|x - y\|^s, \quad \forall\, x, y \in E,\ s \geq 2,\ m_U > 0. \qquad (1.8)$$

Condition (1.7) is called the tangential cone condition and is widely used in the analysis of regularization methods for solving nonlinear ill-posed inverse problems (see [16]).

Note that when $A_i(x) \equiv f_i$ for $i = 1, 2, \ldots, N$, we have $\rho(\alpha) = \|A_0(x_\alpha^\delta) - f_0^\delta\|$. In addition, if $q = 0$, then we obtain the residual principle, investigated in Chapter 3 of [1] and therein references. In the case that $q > 0$, (1.6) is the generalized residual principle, that was first proposed in [11] for linear ill-posed operator equations. Then, it was developed in [12] and [13]. Recently, for nonlinear ill-posed problems involving mappings of monotone type, it was studied in [14, 15], [17]–[20]. So, for the case $A_i(x) \neq f_i$ with $i = 1, 2, \ldots, N$, the principle above is named "generalized quasi-residual one".

2. Main Results

First, we have to prove the following lemmas.

Lemma 2.1. *Let E be a reflexive and strictly convex Banach space with the* ES*-property and strictly convex E^*. Let $\{A_i\}_{i=0}^N$ and $\{f_i\}_{i=0}^N$ be $N+1$ potential, hemicontinuous and monotone mappings on E and $N+1$ elements in E^* such that the set S of solutions for* (1.1) *be nonempty. Then, we have:*

(i) *The function $\rho(\alpha)$, defined in* (1.6), *is continuous on $(\alpha_0, +\infty)$, for each $\alpha_0 > 0$.*
(ii) *If A_N is continuous at x^+ and*

$$\|A_N(x^+) - f_N^\delta\| > 0, \qquad (2.1)$$

for all $\delta \geq 0$, where $f_N^0 = f_N$, then

$$\lim_{\alpha \to +\infty} \rho(\alpha) = +\infty.$$

Proof. From (1.4) it follows

$$\sum_{i=0}^{N} \alpha^{\mu_i}\langle A_i(x_\alpha^\delta) - f_i^\delta, x_\alpha^\delta - z\rangle + \alpha\langle U(x_\alpha^\delta - x^+), x_\alpha^\delta - z\rangle = 0, \quad \forall\, z \in S.$$

Or,

$$\sum_{i=0}^{N}\alpha^{\mu_i}\langle A_i(x_\alpha^\delta)-A_i(z)+A_i(z)-f_i+f_i-f_i^\delta, x_\alpha^\delta-z\rangle + \alpha\langle U(x_\alpha^\delta-x^+), x_\alpha^\delta-z\rangle = 0, \quad \forall z\in S. \tag{2.2}$$

Then, by virtue of (1.2), (2.2) and the monotonicity of A_i, we have

$$\langle U(x_\alpha^\delta-x^+), x_\alpha^\delta-z\rangle \le \delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\|x_\alpha^\delta-z\|, \quad \forall z\in S. \tag{2.3}$$

Therefore,

$$\|x_\alpha^\delta-x^+\|^2-\|x_\alpha^\delta-x^+\|\left[\|z-x^+\|+\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\right]-\|z-x^+\|\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\le 0,$$

and hence,

$$\begin{aligned}0 &\le \|x_\alpha^\delta-x^+\|\\ &\le \frac{1}{2}\Bigg\{\|x^+-z\|+\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\\ &\quad+\sqrt{\left(\|x^+-z\|+\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\right)^2+4\|x^+-z\|\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}}\Bigg\}\\ &\le \|z-x^+\|+\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}+\left(\delta\sum_{i=0}^{N}\frac{1}{\alpha^{1-\mu_i}}\|z-x^+\|\right)^{1/2}.\end{aligned} \tag{2.4}$$

Now, let α and β be any two numbers in $(\alpha_0,+\infty)$. From (1.4), we also have that

$$\sum_{i=0}^{N}\alpha^{\mu_i}(A_i(x_\alpha^\delta)-f_i^\delta)-\sum_{i=0}^{N}\beta^{\mu_i}(A_i(x_\beta^\delta)-f_i^\delta)+\alpha U(x_\alpha^\delta-x^+) - \beta U(x_\beta^\delta-x^+)=0.$$

Consequently,

$$\alpha\langle U(x_\alpha^\delta-x^+)-U(x_\beta^\delta-x^+), x_\alpha^\delta-x_\beta^\delta\rangle+(\alpha-\beta)\langle U(x_\beta^\delta-x^+), x_\alpha^\delta-x_\beta^\delta\rangle + \sum_{i=0}^{N}\alpha^{\mu_i}\langle A_i(x_\alpha^\delta)-A_i(x_\beta^\delta), x_\alpha^\delta-x_\beta^\delta\rangle+\sum_{i=0}^{N}(\alpha^{\mu_i}-\beta^{\mu_i})\langle A_i(x_\beta^\delta)-f_i^\delta, x_\alpha^\delta-x_\beta^\delta\rangle = 0.$$

The last equality together with the following property of U (see Lemma 1.5.4 in [1]),

$$\langle U(x) - U(y), x - y\rangle \geq (\|x\| - \|y\|)^2$$

for any x, $y \in E$, implies that

$$(\|x_\alpha^\delta - x^+\| - \|x_\beta^\delta - x^+\|)^2$$
$$\leq \left[\frac{|\alpha-\beta|}{\alpha_0}\|x_\beta^\delta - x^+\| + \sum_{i=1}^N \frac{|\alpha^{\mu_i} - \beta^{\mu_i}|}{\alpha_0}\|A_i(x_\beta^\delta) - f_i^\delta\|\right](\|x_\alpha^\delta\| + \|x_\beta^\delta\|).$$

So, from the last inequality and (2.4) with α replaced by α_0 in its right-hand side, it follows the continuity of $\|x_\alpha^\delta - x^+\|$ at any $\beta \in (\alpha_0, +\infty)$. Thus, $\rho(\alpha)$ is continuous on $(\alpha_0, +\infty)$. Now, again from (1.4), we can write that

$$\sum_{i=0}^N \alpha^{\mu_i}(A_i(x_\alpha^\delta) - A_i(x^+)) + \alpha U(x_\alpha^\delta - x^+) = \sum_{i=0}^N \alpha^{\mu_i}(f_i^\delta - A_i(x^+)).$$

Acting on the last equality by $x_\alpha^\delta - x^+$ and using the monotonicity of A_i and the definition of U, we obtain that

$$\|x_\alpha^\delta - x^+\| \leq \sum_{i=0}^N \frac{1}{\alpha^{1-\mu_i}}\|f_i^\delta - A_i(x^+)\|.$$

Thus,

$$\lim_{\alpha\to+\infty}\|x_\alpha^\delta - x^+\| = 0.$$

Clearly, the conclusion of the Lemma is followed from the last equality,

$$\rho(\alpha) \geq \alpha^{\mu_N}\left[\|A_N(x_\alpha^\delta) - f_N^\delta\| - \sum_{i=0}^{N-1}\frac{1}{\alpha^{\mu_N-\mu_i}}\|A_i(x_\alpha^\delta) - f_i^\delta\|\right],$$

the continuity of A_N at x^+, the local boundedness of A_i (see [1], Theorem 1.3.16), for $i = 0, 1, \ldots, N$, and $\mu_N > \mu_i$. □

Lemma 2.2. *Let E, A_i and f_i be as in Lemma 2.1. For each p, q, $\delta > 0$, there exists at least a value $\alpha > 0$ such that (1.6) holds.*

Proof. Clearly, from Lemma 2.1, the function $\alpha \to \alpha^{1+q}\|x_\alpha^\delta - x^0\| = \alpha^q\rho(\alpha)$ is continuous on $(\alpha_0, +\infty)$ for any $\alpha_0 > 0$ and

$$\lim_{\alpha\to+\infty}\alpha^q\rho(\alpha) = +\infty.$$

On the other hand, from (2.4) it follows that

$$\alpha^q\rho(\alpha) \leq \alpha^{q+1}\|x^+ - z\| + \alpha^q\delta\sum_{i=0}^N \alpha^{\mu_i} + \alpha^q\left(\alpha\delta\sum_{i=0}^N \alpha^{\mu_i}\|x^+ - z\|\right)^{1/2}.$$

For each $0 < \delta < 1$, we can choose $\alpha > 0$ such that

$$\alpha^{q+1}\|x^+ - z\|,\ \alpha^q\delta\sum_{i=0}^{N}\alpha^{\mu_i},\ \alpha^q\Big(\alpha\delta\sum_{i=0}^{N}\alpha^{\mu_i}\|x^+ - z\|\Big)^{1/2} < \delta^p/3.$$

So, $\alpha^q\rho(\alpha) < \delta^p$ for sufficiently small α. Hence, there exists at least a value $\overline{\alpha} = \alpha(\delta)$ such that $\alpha(\delta)^q\rho(\alpha(\delta)) = \delta^p$. □

Lemma 2.3. *Let E, A_i and f_i be as in Lemma 2.1. Moereover, let any N mappings of the system $\{A_i\}_{i=0}^N$ be strictly monotone at x^+. Then,*

$$\lim_{\delta\to 0}\alpha(\delta) = 0.$$

Proof. Without any loss of generality, we assume that A_i is a strictly monotone mapping at x^+ with $i = 0, 1, \ldots, N-1$. We shall prove by supposing that the conclusion is not true. Then, there is a sequence $\delta_k \to 0$ as $k \to +\infty$ with
1) $\overline{\alpha}_k = \alpha(\delta_k) \to C_0$, some positive constant; or
2) $\overline{\alpha}_k \to +\infty$.

In the case 1), from (1.6), is follows that $C_0^{1+q}\lim_{k\to+\infty}\|x_{\overline{\alpha}_k}^{\delta_k} - x^+\| = 0$. Next, replacing δ, α and x in (1.4), respectively, by δ_k, $\overline{\alpha}_k$ and $x_{\overline{\alpha}_k}^{\delta_k}$, and passing $k \to +\infty$, we obtain that

$$\sum_{i=0}^{N} C_0^{\mu_i}(A_i(x^+) - A_i(z)) = 0, \quad z \in S. \tag{2.5}$$

Acting on the equality by $x^+ - z$ and using the monotonicity of A_i for $i = 0, 1, \ldots, N$, and $C_0 > 0$, we have

$$\langle A_i(x^+) - A_i(z), x^+ - z\rangle = 0, \quad i = 0, 1, \ldots, N.$$

Since A_i is strictly monotone at x^+ for $i = 0, 1, \ldots, N-1$, $x^+ \in \bigcap_{i=0}^{N-1} S_i$. Therefore, from (2.5) it follows that $x^+ \in S_N$. Hence, $x^+ \in S$, that contradicts the assumption $x^+ \notin S$.

In the case 2), also from (1.6), it follows that

$$\lim_{k\to+\infty}\|x_{\overline{\alpha}_k}^{\delta_k} - x^+\| = \lim_{k\to+\infty}\frac{\rho(\overline{\alpha}_k)}{\overline{\alpha}_k} = \lim_{k\to+\infty}\frac{\delta_k^p}{\overline{\alpha}_k^{1+q}} = 0. \tag{2.6}$$

Again, replacing δ, α and x in (1.4), respectively, by δ_k, $\overline{\alpha}_k$ and $x_{\overline{\alpha}_k}^{\delta_k}$, we obtain that

$$\overline{\alpha}_k^{\mu_N}\Big[\|A_N(x_{\overline{\alpha}_k}^{\delta_k}) - f_N^{\delta_k}\| - \sum_{i=0}^{N-1}\frac{1}{\overline{\alpha}_k{}^{\mu_N-\mu_i}}\|A_i(x_{\overline{\alpha}_k}^{\delta_k}) - f_i^{\delta_k}\|\Big] - \|A_0(x_{\overline{\alpha}_k}^{\delta_k}) - f_0^{\delta_k}\|$$
$$\le \overline{\alpha}_k\|x_{\overline{\alpha}_k}^{\delta_k} - x^+\| = \rho(\overline{\alpha}_k) = \overline{\alpha}_k^{-q}\delta_k^p.$$

Tending $k \to +\infty$ in the last inequality and using (2.6), the local boundedness of A_i, for $i = 0, 1, \dots, N-1$, the continuity of A_N at x^+ with condition (2.1), and the fact that $\overline{\alpha}_k \to +\infty$ and $\delta_k \to 0$, we obtain the inequality $+\infty \le 0$, that is impossible. This completes the proof. □

Lemma 2.4. *Let E, A_i and f_i be as in Lemma 2.3. If $q \ge p$, then*

$$\lim_{\delta \to 0} \delta/\alpha(\delta) = 0.$$

Proof. It is easy to see that

$$\left[\frac{\delta}{\alpha(\delta)}\right]^p = [\delta^p \alpha(\delta)^{-q}]\alpha(\delta)^{q-p} = \rho(\alpha(\delta))\alpha(\delta)^{q-p}.$$

On the other hand, from (2.4) it follows that

$$\rho(\alpha(\delta)) \le \alpha(\delta)\|x^+ - z\| + \delta\sum_{i=0}^{N} \alpha^{\mu_i}(\delta) + \left(\alpha(\delta)\delta\sum_{i=0}^{N} \alpha^{\mu_i}(\delta)\|x^+ - z\|\right)^{1/2}.$$

Therefore,

$$\lim_{\delta \to 0}\left[\frac{\delta}{\alpha(\delta)}\right]^p = 0.$$

The lemma is proved. □

Lemma 2.5. *Let E, A_i and f_i be as in Lemma 2.3. If $0 < p \le q$, then*

$$\lim_{\delta \to 0} x^\delta_{\alpha(\delta)} = x_0.$$

Proof. It follows from Lemmas 2.3, 2.4 and standard results about convergence of the Browder–Tikhonov regularization method for (1.4) (see [8, 20]). □

Lemma 2.6. *Let E, A_i and f_i be as in Lemma 2.3 and let $0 < p \le q$. Then, there are constants C_1, $C_2 > 0$ such that, for sufficiently small $\delta > 0$, the relation*

$$C_1 \le \delta^p \alpha^{-1-q}(\delta) \le C_2$$

holds.

Proof. Because of (1.2) and (1.5), we have, for all $\alpha > 0$, $f_i^\delta \in E^*$,

$$\rho(\alpha) = \alpha(\delta)\|x^\delta_{\alpha(\delta)} - x^+\|,$$

which together with Lemma 2.5 implies that

$$\lim_{\delta \to 0} \delta^p \alpha^{-1-q}(\delta) = \lim_{\delta \to 0} \alpha^{-1}(\delta)\rho(\alpha(\delta))) = \|x_0 - x^+\| > 0.$$

This implies the conclusion of the lemma. □

Theorem 2.7. *Let E, A_i and f_i be as in Lemma 2.3. In addition, assume that the following conditions hold:*

(i) *the duality mapping U satisfies (1.8);*

(ii) *A_0 is Fréchet differentiable at some neighbourhood of S with (1.7);*

(iii) *there exists an element $\omega \in E$ such that*

$$A_0'(x_0)^*\omega = U(x_0 - x^+), \text{ and}$$

(iv) *the parameter $\alpha = \alpha(\delta)$ is chosen by (1.6) with $q > p$.*

Then, we have

$$\|x^\delta_{\alpha(\delta)} - x_0\| = O(\delta^\eta), \quad \eta = \frac{1}{1+q}\min\left\{(q-p)/(s-1);\ p\mu_1/s\right\}.$$

Proof. From (1.4), (1.8), the monotonicity of A_i and condition (iii) of the theorem it follows

$$\begin{aligned}
m_U\|x^\delta_\alpha - x_0\|^s &\le \langle U(x^\delta_\alpha - x^+) - U(x_0 - x^+), x^\delta_\alpha - x_0\rangle \\
&= \frac{1}{\alpha}\sum_{i=0}^N \alpha^{\mu_i}\langle f_i^\delta - A_i(x^\delta_\alpha), x^\delta_\alpha - x_0\rangle \\
&\quad + \langle U(x_0 - x^+), x_0 - x^\delta_\alpha\rangle \\
&\le \frac{\delta}{\alpha}\sum_{i=0}^N \alpha^{\mu_i}\|x^\delta_\alpha - x_0\| + \langle \omega, A_0'(x_0)(x_0 - x^\delta_\alpha)\rangle \\
&\le \frac{\delta}{\alpha}\sum_{i=0}^N \alpha^{\mu_i}\|x^\delta_\alpha - x_0\| + \|\omega\|\|A_0'(x_0)(x_0 - x^\delta_\alpha)\|.
\end{aligned} \tag{2.7}$$

On the other hand, from (1.7), we have that

$$\begin{aligned}
&\|A_0'(x_0)(x_0 - x^\delta_\alpha)\| \\
&\le (1+\tau)\|A_0(x^\delta_\alpha) - f_0\| \le (1+\tau)\left[\|A_0(x^\delta_\alpha) - f_0^\delta\| + \delta\right] \\
&\le (1+\tau)\left[\delta + \sum_{i=1}^N \alpha^{\mu_i}\|A_i(x^\delta_\alpha) - f_i^\delta\| + \alpha\|x^\delta_\alpha - x^+\|\right] \\
&\le (1+\tau)\left[\delta\sum_{i=0}^N \alpha^{\mu_i} + \alpha\|x^\delta_\alpha - x^+\| + \sum_{i=1}^N \alpha^{\mu_i}\|A_i(x^\delta_\alpha) - A_i(x_0)\|\right].
\end{aligned}$$

If α is chosen by (1.6), then $\|x^\delta_{\alpha(\delta)} - x_0\| < c$, a sufficiently small and positive constant, for sufficiently small δ, and $\alpha(\delta) \le 1$. Consequently, we have that $\alpha^{\mu_i}(\delta) \le \alpha^{\mu_1}(\delta)$ and $\|A_i(x_{\alpha(\delta)}) - A_i(x_0)\| \le C$, a positive constant, because

A_i is locally bounded at x_0. Therefore, from (2.7) and Lemma 2.6, we obtain that

$$m_U\|x^\delta_{\alpha(\delta)} - x_0\|^s \leq (1+N)C_2\delta^{1-p}\alpha^q(\delta)\|x^\delta_{\alpha(\delta)} - x_0\| + \|\omega\|(1+\tau)\left[\delta(1+N) + \alpha^{-q}(\delta)\delta^p + CN\alpha^{\mu_1}(\delta)\right]$$

$$\leq (1+N)C_2C_1^{-q/(1+q)}\delta^{\frac{1-p}{1+q}}\|x^\delta_{\alpha(\delta)} - x_0\| + CNC_1^{-\frac{\mu_1}{1+q}}\delta^{\frac{p\mu_1}{1+q}}.$$

Using the implication

$$a,\ b,\ c\ \geq\ 0,\ p\ >\ q,\ a^p \leq ba^q + c \implies a^p = O(b^{p/(p-q)} + c)$$

we obtain

$$\|x^\delta_{\alpha(\delta)} - x_0\| = O(\delta^\eta). \Longrightarrow\ a^p = O(b^{p/(p-q)} + c)$$

The theorem is proved. □

Acknowledgments: This research is founded by Vietnamese National Foundation of Science and Technology Development.

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