



# MATH TOOLKIT FOR REAL-TIME DEVELOPMENT

Jack W. Crenshaw

# **Math Toolkit for Real-Time Development**

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