



# MATH TOOLKIT FOR REAL-TIME DEVELOPMENT

Jack W. Crenshaw

# **Math Toolkit for Real-Time Development**

*Jack W. Crenshaw*

CMP Books  
Lawrence, Kansas 66046

**CMP Books  
CMP Media, Inc.  
1601 W. 23rd Street, Suite 200  
Lawrence, KS 66046  
USA**

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Cover art created by Janet Phares.

***Distributed in the U.S. by:***  
**Publishers Group West**  
**1700 Fourth Street**  
**Berkeley, California 94710**  
**1-800-788-3123**  
**[www.pgw.com](http://www.pgw.com)**

***Distributed in Canada by:***  
**Jaguar Book Group**  
**100 Armstrong Avenue**  
**Georgetown, Ontario M6K 3E7 Canada**  
**905-877-4483**

**ISBN: 1-929629-09-5**

**CMP** *Books*

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