THE CONVERGENCE OF THE REGULARIZED SOLUTION FOR ILL-POSED SYSTEM OF OPERATOR EQUATIONS

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Abstract: The purpose of this paper is to study the convergence of regularization solutions for ill-posed system of operator equations on real Hilbert spaces.
[1]

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. We are concerned with the following vector optimization problem of finding an element $x^0 \in H$ such that

$$\varphi_j(x^0) = \inf_{x \in H} \varphi_j(x), \quad \forall j = 0, 1, ..., N, \tag{1.1}$$

where $\varphi_j(x) = \frac{1}{2}\langle A_j x, x\rangle - \langle x, f_j\rangle$ are quadratic functions, each A_j is a positive self-adjoint bounded linear operator on H, $f_j \in H$, $j = 0, 1, ..., N$, $N \geq 0$ is an integer.

Set

$$S_j = \{\bar{x} \in H : \varphi_j(\bar{x}) = \inf_{x \in H} \varphi_j(x)\}, \quad j = 0, 1, ..., N$$
$$S = \cap_{j=0}^{N} S_j.$$

Here, we suppose that, $S \neq \emptyset$. Because of the fact that $\varphi'_j(x) = A_j x - f_j$, each S_j, $j = 0, 1, \ldots, N$, coincides with the set of solutions of the following operator equation

$$A_j(x) = f_j, \tag{1.2}$$

and it is a closed convex subset in H (see [4, 6]).

Without additional conditions on A_j such as the strongly or uniformly monotone property, each equation in (1.2) is ill-posed. By this, we mean that the solution set S_j does not depend continuously on the data A_j and f_j. Therefore, to find a solution of each equation in (1.2), we have to use stable methods. One of those methods is the Tikhonov regularization in the form (see [1])

$$A_j^h(x) + \alpha U^s(x - x_*) = f_j^\delta, \tag{1.3}$$

[1]Key words: Monotone operators, hemi-continuous, Gâteaux and Fréchet differentiable, quadratic optimization, Tikhonov regularization.

where $\alpha > 0$ is the small parameter of regularization, $U^s : H \to H$ is the dual mapping of H satisfying the condition

$$\langle U^s(x), x\rangle = \|x\|^s, \quad \|U^s(x)\| = \|x\|^{s-1},$$

A_j^h are the hemicontinuous monotone operators, (A_j^h, f_j^h) are approximation for (A_j, f_j) in the sense that

$$\|A_j^h(x) - A_j(x)\| \le hg(\|x\|), \quad \|f_j^\delta - f_j\| \le \delta \tag{1.4}$$

with levels $(h, \delta) \to 0$ and $g(t)$ is the bounded nonegative function $t \ge 0$, x_* is some element in H playing the role of criterion selection. By the choice of x_* we can influence the solution we want to approximate.
For each j, equation (1.3) has a unique solution $x_j^{\alpha,\tau}$, $\tau = (h, \delta)$ and if h/α, δ/α, $\alpha \to 0$ then $x_j^{\alpha,\tau} \to x_j \in S_j$ with x_*-minimal norm (see [1]), i.e.,

$$\|x_j - x_*\| = \min_{x\in S_j} \|x - x_*\|, \quad j = 0, 1, ..., N.$$

The problem to be considered in this paper is that of finding x_α^τ such that x_α^τ converges to $x^0 \in S$ and study the convergence of sequences $\{x_\alpha^\tau\}$ to x^0.
We assume that U^s satisfies the condition

$$\langle U^s(x) - U^s(y), x - y\rangle \ge m_U\|x - y\|^s, \quad m_U > 0, \quad s \ge 2, \quad \forall x, y \in H. \tag{1.5}$$

Hereafter the symbols $\rightharpoonup$ and $\to$ indicate weak convergence and convergence in norm, respectively, while the notation $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULT

The first, we consider the operator equation

$$\sum_{j=0}^{N} \alpha^{\lambda_j}(A_j^h(x) - f_j^\delta) + \alpha U^s(x - x_*) = \theta, \tag{2.1}$$

$$\lambda_0 = 0 < \lambda_j < \lambda_{j+1} < 1, \quad j = 1, 2, ..., N - 1.$$

Since each A_j^h is maximal monotone, $\sum_{j=0}^{N} \alpha^{\lambda_j} A_j^h + \alpha U^s$ is maximal monotone and coercitive (see [1, 2, 3]). Hence, equation (2.1) has a unique solution, which is denoted by x_α^τ.

The convergence of $\{x_\alpha^\tau\}$ to x^0 is determined by the following theorem.

Theorem 2.1. *If h/α, δ/α, $\alpha \to 0$ then the sequence x_α^τ converges to x^0.*

Proof. For $x \in S$, it follows from (1.2) and (2.1) that

$$\sum_{j=0}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - f_j^\delta - A_j(x) + f_j, x_\alpha^\tau - x\rangle + \alpha\langle U^s(x_\alpha^\tau - x_*) - U^s(x - x_*), x_\alpha^\tau - x\rangle = \alpha\langle U^s(x - x_*), x - x_\alpha^\tau\rangle.$$

On the basis of (1.5) and the monotonicity of A_j^h we have

$$\alpha m_U \|x_\alpha^\tau - x\|^s \leq \sum_{j=0}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - A_j^h(x) + A_j^h(x) - A_j(x) + f_j - f_j^\delta, x - x_\alpha^\tau\rangle + \alpha\langle U^s(x - x_*), x - x_\alpha^\tau\rangle,$$

or

$$m_U \|x_\alpha^\tau - x\|^s \leq \frac{1}{\alpha}(N+1)(hg(\|x\|) + \delta)\|x - x_\alpha^\tau\| + \langle U^s(x - x_*), x - x_\alpha^\tau\rangle. \tag{2.2}$$

This inequality gives us the boundedness of the sequence $\{x_\alpha^\tau\}$. Then, there exists a subsequence of the sequence $\{x_\alpha^\tau\}$ converging weakly to $\hat{x}$ in H. Without loss of generality, we assume that $x_\alpha^\tau \rightharpoonup \hat{x}$ as h/α, δ/α and $\alpha \to 0$.

First, we prove that $\hat{x} \in S_0$. Indeed, by virtue of the monotonicity of A_j^h, U^s and (2.1) we have

$$\begin{aligned}\langle A_0^h(x) - f_0^\delta, x - x_\alpha^\tau\rangle &\geq \langle A_0^h(x_\alpha^\tau) - f_0^\delta, x - x_\alpha^\tau\rangle \\ &= \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - f_j^\delta, x_\alpha^\tau - x\rangle + \alpha\langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x\rangle \\ &\geq \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x) - f_j^\delta, x_\alpha^\tau - x\rangle + \alpha\langle U^s(x - x_*), x_\alpha^\tau - x\rangle, \quad \forall x \in H.\end{aligned}$$

By letting $h, \delta, \alpha \to 0$ in this inequality we obtain

$$\langle A_0(x) - f_0, x - \hat{x}\rangle \geq 0, \quad \forall x \in H.$$

Consequently, $\hat{x} \in S_0$ (see [6]).

Now, we shall prove that $\hat{x} \in S_j$, $j = 1, 2..., N$. Indeed, by (2.1) and making use of the monotonicity of A_j^h, it follows that

$$\begin{aligned}
&\langle A_1^h(x_\alpha^\tau) - f_1^\delta, x_\alpha^\tau - x\rangle + \sum_{j=2}^{N} \alpha^{\lambda_j-\lambda_1}\langle A_j^h(x_\alpha^\tau) - f_j^\delta, x_\alpha^\tau - x\rangle \\
&\quad + \alpha^{1-\lambda_1}\langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x\rangle \\
&\quad = \frac{1}{\alpha^{\lambda_1}}\langle A_0^h(x_\alpha^\tau) - A_0^h(x) + A_0^h(x) - A_0(x) + f_0 - f_0^\delta, x - x_\alpha^\tau\rangle \\
&\quad \le \frac{\alpha^{1-\lambda_1}}{\alpha}(hg(\|x\|)+\delta)\|x - x_\alpha^\tau\|, \quad \forall x \in S_0
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle A_1^h(x) - f_1^\delta, x_\alpha^\tau - x\rangle + \sum_{j=2}^{N} \alpha^{\lambda_j-\lambda_1}\langle A_j^h(x) - f_j^\delta, x_\alpha^\tau - x\rangle \\
&\qquad + \alpha^{1-\lambda_1}\langle U^s(x - x_*), x_\alpha^\tau - x\rangle \\
&\qquad \le \frac{\alpha^{1-\lambda_1}}{\alpha}(hg(\|x\|)+\delta)\|x - x_\alpha^\tau\|, \quad \forall x \in S_0.
\end{aligned}$$

After passing $h, \delta, \alpha \to 0$, we obtain

$$\langle A_1(x) - f_1, \hat{x} - x\rangle \le 0, \quad \forall x \in S_0.$$

Thus, $\hat{x}$ is a local minimizer for φ_1 on S_0 (see [5]). Since $S_0 \cap S_1 \neq \emptyset$, then, $\hat{x}$ is also a global minimizer for φ_1, i.e., $\hat{x} \in S_1$.

Set $\tilde{S}_i = \cap_{k=0}^{i} S_k$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \emptyset$. Now, suppose that we have proved $\hat{x} \in \tilde{S}_i$ and we need to show that $\hat{x}$ belongs to S_{i+1}. Again, by virtue of (2.1) for $x \in \tilde{S}_i$, we can write

$$\begin{aligned}
&\langle A_{i+1}^h(x) - f_{i+1}^\delta, x_\alpha^\tau - x\rangle + \sum_{j=i+2}^{N} \alpha^{\lambda_j-\lambda_{i+1}}\langle A_j^h(x_\alpha^\tau) - f_j^\delta, x_\alpha^\tau - x\rangle \\
&\qquad + \alpha^{1-\lambda_{i+1}}\langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x\rangle \\
&\qquad \le \sum_{k=0}^{i} \alpha^{\lambda_k-\lambda_{i+1}}\langle A_k^h(x_\alpha^\tau) - f_k^\delta, x - x_\alpha^\tau\rangle \\
&\qquad \le \frac{1}{\alpha}\sum_{k=0}^{i} \alpha^{\lambda_k+1-\lambda_{i+1}}\langle A_k^h(x) - A_k(x) + f_k - f_k^\delta, x - x_\alpha^\tau\rangle \\
&\qquad \le \frac{1}{\alpha}(N+1)\big(hg(\|x\|)+\delta\big)\|x - x_\alpha^\tau\|.
\end{aligned}$$

Therefore, by letting $h, \delta, \alpha \to 0$, we have

$$\langle A_{i+1}(x) - f_{i+1}, \hat{x} - x\rangle \leq 0, \quad \forall x \in \tilde{S}_i.$$

As a result, $\hat{x} \in S_{i+1}$.

On the other hand, it follows from (2.2) that

$$\langle U^s(x - x_*), x - \hat{x}\rangle \geq 0, \quad \forall x \in S.$$

Since S_j is closed convex, S is also closed convex. Replacing x by $t\hat{x} + (1-t)x$, $t \in (0,1)$ in the last inequality, and dividing by $(1-t)$ and letting t to 1, we obtain

$$\langle U^s(\hat{x} - x_*), x - \hat{x}\rangle \geq 0, \quad \forall x \in S.$$

Hence $\|\hat{x} - x_*\| \leq \|x - x_*\|, \forall x \in S$. Because of the convexity and the closedness of S, and the strictly convexity of H we deduce that $\hat{x} = x^0$. So, every sequence $\{x_\alpha^\tau\}$ converges weakly to x^0. It follows from (2.1) that the sequence $\{x_\alpha^\tau\}$ converges strongly to x^0.

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