Relative formation control of mobile agents

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Abstract

A constructive method is presented to design bounded and continuous cooperative controllers that force a group of N mobile agents with limited sensing ranges to stabilize at a desired location, and guarantee no collisions between the agents. The control development is based on new general potential functions, which attain the minimum value when the desired formation is achieved, are equal to infinity when a collision occurs, and are continuous at switches. The multiple Lyapunov function (MLF) approach is used to analyze stability of the closed loop switched system.

Index Terms

Formation stabilization, bounded control, multiple Lyapunov function, switched system.

I. Introduction

Technological advances in communication systems and the growing ease in making small, low power and inexpensive mobile agents make it possible to deploy a group of networked mobile vehicles to offer potential advantages in performance, redundancy, fault tolerance, and robustness. Formation control of multiple agents has received a lot of attention from both robotics and control communities. Basically, formation control involves the control of positions of a group of the agents such that they stabilize/track desired locations relative to reference point(s), which can be another agent(s) within the team, and can either be stationary or moving. Three popular approaches to formation control are leader-following (e.g. [1], [2]), behavioral (e.g. [3], [4]), and use of virtual structures (e.g. [5], [6]). Most research works investigating formation control utilize one or more of these approaches in either a centralized or decentralized manner. Centralized control schemes, see e.g. [2] and [7], use a single controller that generates collision free trajectories in the workspace. Although these guarantee a complete solution, centralized schemes require high computational power and are not robust due to the heavy dependence on a single controller. On the other hand, decentralized schemes, see e.g. [8], [9] and [10], require less computational effort, and are relatively more scalable to the team size. The decentralized approach usually involves a combination of agent based local potential fields ([2], [10], [11]. The main problem with the decentralized approach, when collision avoidance is taken into account, is that it is extremely difficult to predict and control the critical points (the controlled system often has multiple equilibrium points). It is difficult to design a controller such that all the equilibrium points except for the desired equilibrium ones are unstable points. Recently, a method based on a different navigation function from [12] provided a centralized formation stabilization control design strategy is proposed in [9]. This work is extended to a decentralized version in [13]. However, the navigation function approaches a finite value when a collision occurs, and the formation is stabilized to any point in workspace instead of being ”tied” to a fixed coordinate frame. In [14], [12], [9] and [13], the tuning constants, which are crucial to guarantee that the only desired equilibrium points are asymptotic stable and that the other critical points are unstable, cannot be obtained explicitly but ”are chosen sufficiently small”. When it comes to a practical implementation, an important issue is ”how small these constants should be?” Moreover, the

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control design methods ([2], [15], [10]) based on the potential/navigation functions that are equal to infinity when a collision occurs exhibit very large control efforts if the agents are close to each other. Hence, a bounded control is called for. These problems motivate the work in this paper.

In this paper, we design bounded and continuous cooperative controllers for formation stabilization of a group of mobile agents with limited sensing ranges. New general continuous potential functions are constructed to design the controllers that yield (almost) global asymptotic convergence of a group of mobile agents to a desired formation, and guarantee no collisions among the agents. Continuity of the potential functions at switches significantly simplifies stability analysis of the controlled (switched) system based on the MLF approach [19]. Moreover, the controlled system exhibits multiple equilibrium points due to collision avoidance taken into account. We therefore investigate the behavior of equilibrium points by linearizing the closed loop system around those points, and show that critical points, other than the desired point for an agent, are unstable.

II. Problem statement

We consider a group of N mobile agents, of which each has the following dynamics

$$\dot{q}_i = u_i, i = 1, ..., N \tag{1}$$

where $q_i \in \mathbb{R}^n$ and $u_i \in D \subset \mathbb{R}^n$ are the state and control input of the agent i. We assume that $n > 1$ and $N > 1$. In this paper, we treat each agent as an autonomous point. This assumption is not as restrictive as it may seem since various shapes can be mapped to single points through a series of transformations as shown in seminal papers [14], [12], [7]. Our task is to design the bounded control input u_i for each agent i that forces the group of N agents to stabilize with respect to their group members in configurations that make a particular formation specified by a desired vector $\bar{l}(\eta) = [l_{12}^T(\eta), l_{13}^T(\eta), ..., l_{1N}^T(\eta), l_{23}^T(\eta), l_{24}^T(\eta), ..., l_{2N}^T(\eta), ..., l_{N-1,N}^T(\eta)]^T$, where η is the formation parameter vector, while avoiding collisions between themselves. The parameter vector η is used to specify rotation, expansion and contraction of the formation such that when η converges to its desired value η_f, the desired shape of the formation is achieved. In addition, it requires all the agents align their velocity vectors to a desired bounded one u_d , and move toward specified directions specified by the desired formation velocity vector u_d. Finally, the collision avoidance between the agents are to be taken into account only when they are in their proximity to eliminate unnecessary control effort. The control objective is formally stated as follows:

Control objective: Assume that at the initial time t_0 each agent initializes at a different location, and that each agent has a different desired location, i.e. there exist strictly positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, and a nonnegative constant u_d^M such that for all $(i, j) \in \{1, 2, ..., N\}, i \neq j, t \geq t_0 \geq 0$:

$$\|q_i(t_0) - q_j(t_0)\| \geq \epsilon_1, \quad \|l_{ij}(\eta(t)\| \geq \epsilon_2, \quad \left\|\frac{\partial l_{ij}(\eta)}{\partial \eta}\right\| \leq \epsilon_3, \quad \|u_d(t)\| \leq u_d^M. \tag{2}$$

Design the bounded control input u_i for each agent i , and an update law for the formation parameter vector η such that each agent (almost) globally asymptotically approaches its desired location to form a desired formation, and that the agents' velocity converges to the desired (bounded) velocity u_d while avoiding collisions with all other agents in the group, i.e. for all $(i,j) \in \{1,2,...,N\}, i \neq j, t \geq t_0 \geq 0$:

$$\begin{array}{l} \lim_{t\to\infty}(q_i(t) - q_j(t) - l_{ij}) = 0, \\ \|q_i(t) - q_j(t)\| \geq \epsilon_4, \\ \lim_{t\to\infty}(u_i(t) - u_d) = 0, \\ \lim_{t\to\infty}(\eta(t) - \eta_f) = 0, \\ \|u_i(t)\| \leq u_i^M \end{array} \quad (3)$$

where u_i^M and ϵ_4 are strictly positive constants. The constant u_i^M is such that $u_i^M \geq u_d^M + \epsilon_5$ with ϵ_5 being a strictly positive constant. Moreover, the control u_i takes the collision avoidance with other agents in the group into account only when these agents are in the proximity of the agent i, i.e. in a sphere, which is centered at the agent i and has a radius of R_i.

III. Control design

For each agent i, we consider the following potential function

$$\varphi_i = \gamma_i + \beta_i + \frac{1}{2}\|\eta - \eta_f\|^2 \quad (4)$$

where γ_i and β_i are the goal and related collision avoidance functions for the agent i specified as follows:

-The goal function γ_i is designed such that it puts penalty on the stabilization error for the agent i, and is equal to zero when the agent is at its desired position with respect to all other agents in the group. A simple choice of this function is

$$\gamma_i = \sum_{j\in\mathbb{N}_i} \|q_{ij} - l_{ij}\|^2 \quad (5)$$

where $q_{ij} = q_i - q_j$ and $\mathbb{N}_i$ is the set that contains all the agents in the group except for the agent i.

-The related collision function β_i should be chosen such that it is equal to infinity whenever any agents come in contact with the agent i , i.e. a collision occurs, and attains the minimum value when the agent i is at its desired location with respect to other group members belong to the set $\mathbb{N}_i$ agents. This function is chosen as follows:

$$\beta_i = \sum_{j\in\mathbb{N}_i} (a_{ij}\beta_{ij}) \quad (6)$$

where a_{ij} is the switching parameter, which determines the collisions between the agent i and the agent j are taken into account only if they are in their proximities. The switching rule for a_{ij} is specified as:

$$\begin{array}{lll} a_{ij} = 1 & \text{if} & \|q_{ij}\| \leq R_{ij}, \\ a_{ij} = 0 & \text{if} & \|q_{ij}\| > R_{ij} \end{array} \quad (7)$$

where R_{ij} is a strictly positive constant, and is such that $R_{ij} \leq min(R_i, R_j)$. The function β_{ij} is a function of $\|q_{ij}\|^2/2$, $\|l_{ij}\|^2/2$ and $R_{ij}^2/2$, and enjoys the following properties:

$$\begin{array}{ll} 1) & \beta_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| = \|l_{ij}\| \quad \text{or} \quad \|q_{ij}\| = R_{ij}, \\ 2) & \beta_{ij} > 0 \quad \text{if} \quad \|q_{ij}\| \neq \|l_{ij}\| \quad \text{and} \quad \|q_{ij}\| \neq R_{ij}, \\ 3) & \beta_{ij}\big|_{\|q_{ij}\|=0} = \infty, \\ 4) & \beta'_{ij}\big|_{\|q_{ij}\|=\|l_{ij}\|} = 0, \beta''_{ij}\big|_{\|q_{ij}\|=\|l_{ij}\|} \geq 0, \\ 5) & \left\|\dfrac{\partial\beta_{ij}}{\partial(\|l_{ij}\|^2/2)} l_{ij}\right\| \leq \epsilon_5\beta_{ij} + \epsilon_6 \end{array} \quad (8)$$

where $\beta'_{ij} = \partial\beta_{ij}/\partial(\|q_{ij}\|^2/2)$ and $\beta''_{ij} = \partial^2\beta_{ij}/\partial(\|q_{ij}\|^2/2)^2$, ϵ_5 and ϵ_6 are nonnegative constants. It is noted that $\beta_{ij} = \beta_{ji}$.

Remark 1: Properties 1), 2) and 3) of β_{ij} imply that the function β_i (so the function φ_i defined in (4) with the function γ_i given in (5)) is positive definite and is equal to infinity when a collision between any agents in the group occurs. In addition, Properties 1) and 2) of β_{ij} ensure that the function φ_i is continuous when the switching parameter a_{ij} is switched according to the switching rule (7). As it will be seen in the proof of Theorem 1, continuity of φ_i significantly simplifies stability analysis of the closed loop switched system using the MLF approach. Property 4) of β_{ij} and the function γ_i given in (5) ensure that the function attains the (unique) minimum value of zero when all the agents are at their desired positions. Property 5) of β_{ij} is needed to prove existence of the solutions of the closed loop system (see proof of Theorem 1).

There are many functions that satisfy all Properties 1)-5) given in (8) such as

$$\beta_{ij} = \left(\frac{\|q_{ij}\|^2/2}{(\|l_{ij}\|^2/2)^2} + \frac{1}{(\|q_{ij}\|^2/2)} - \frac{2}{(\|l_{ij}\|^2/2)}\right)(\|q_{ij}\|^2/2 - R_{ij}^2/2)^2 \quad (9)$$

and

$$\beta_{ij} = \frac{(\|q_{ij}\|^2/2 - \|l_{ij}\|^2/2)^2}{(\|q_{ij}\|^2/2)}(\|q_{ij}\|^2/2 - R_{ij}^2/2)^2. \quad (10)$$

The derivative of φ_i between the switching intervals (since the function φ_i is continuous (though nonsmooth), the gradient $\nabla_{q_{ij}}\varphi_i$ is empty at $\|q_{ij}\| = R_{ij}$, see ([10])), along the solutions of (1) satisfies

$$\dot{\varphi}_i = \sum_{j\in\mathbb{N}_i}(q_{ij} - l_{ij} + a_{ij}\beta'_{ij}q_{ij})^T(u_i - u_j) + \Phi_i\dot{\eta} + (\eta - \eta_f)^T\dot{\eta} \quad (11)$$

where

$$\Phi_i = \sum_{j\in\mathbb{N}_i}\big[-(q_{ij} - l_{ij}) + a_{ij}\frac{\partial\beta_{ij}}{\partial(\|l_{ij}\|^2/2)}l_{ij}\big]^T\frac{\partial l_{ij}}{\partial\eta}. \quad (12)$$

Adding and subtracting u_d to $(u_i - u_j)$ in the right hand side of (11) results in

$$\begin{aligned}\dot{\varphi}_i &= \sum_{j\in\mathbb{N}_i}(q_{ij} - l_{ij} + a_{ij}\beta'_{ij}q_{ij})^T(u_i - u_d - (u_j - u_d)) + \Phi_i\dot{\eta} + (\eta - \eta_f)^T\dot{\eta} \\ &= \Omega_i^T(u_i - u_d) - \sum_{j\in\mathbb{N}_i}(q_{ij} - l_{ij} + a_{ij}\beta'_{ij}q_{ij})^T(u_j - u_d) + \Phi_i\dot{\eta} + (\eta - \eta_f)^T\dot{\eta} \quad (13)\end{aligned}$$

where

$$\Omega_i = \sum_{j\in\mathbb{N}_i}(q_{ij} - l_{ij} + a_{ij}\beta'_{ij}q_{ij}). \quad (14)$$

From (13), we simply choose the control u_i and the update law for η as follows

$$\begin{aligned}u_i &= -C\Psi_i(\Omega_i) + u_d, \\ \dot{\eta} &= -\Gamma(\eta - \eta_f) \quad (15)\end{aligned}$$

where $C = I_{n\times n}c$ with $I_{n\times n}$ being the n dimensional identity matrix and c being a positive constant, Γ is a symmetric positive definite matrix, and $\Psi_i(\Omega_i)$ denotes a vector of bounded functions of elements of Ω_i in the sense that

$$\Psi_i(\Omega_i) = \big[\psi_i(\Omega_i^1)\ \psi_i(\Omega_i^2), ..., \psi_i(\Omega_i^h), ..., \psi_(\Omega_i^n)\big]^T \quad (16)$$

where Ω_i^h is the h^{th} element of Ω_i, i.e. $\Omega_i = [\Omega_i^1\ \Omega_i^2...\Omega_i^h...\Omega_i^n]^T$. The function $\psi_i(\Omega_i^h)$ is a smooth, class-K, and bounded function of Ω_i^h, which satisfies the following properties

$$\begin{aligned}
&1) \quad |\psi_i(\Omega_i^h)| \leq \psi_i^M, \forall\, \Omega_i^h \in \mathbb{R}, \\
&2) \quad \psi_i(\Omega_i^h) = 0 \quad \text{if} \quad \Omega_i^h = 0, \\
&3) \quad \Omega_i^h \psi_i(\Omega_i^h) > 0 \quad \text{if} \quad \Omega_i^h \neq 0, \\
&4) \quad \partial\psi_i(\Omega_i^h)/\partial\Omega_i^h\big|_{\Omega_i^h=0} = 1
\end{aligned} \tag{17}$$

where ψ_i^M is a strictly positive constant. Some functions that satisfy all properties listed in (17) are $\arctan(\Omega_i^h)$, $\tanh(\Omega_i^h)$, and $\Omega_i^h/\sqrt{1+\Omega_i^h}$. Indeed, the control u_i is a bounded one in the sense that $\|u_i(t)\| \leq c\sqrt{\sum_{i=1}^n(\psi_i^M)^2} + u_d^M := u_i^M, \forall t \geq t_0 \geq 0$.

Remark 2: When Ω_i defined in (14) is substituted into the control u_i in (15) and the negative sign is moved to inside of the bounding function Ψ_i, we can see that the argument of the bounding function of the h^{th} element of the control consists of two parts: $-\sum_{j\in\mathbb{N}_i}(q_{ij}^h - l_{ij}^h)$ and $-\sum_{j\in\mathbb{N}_i}(a_{ij}\beta'_{ij}q_{ij}^h)$ with q_{ij}^h and l_{ij}^h being the h^{th} elements of q_{ij} and l_{ij}. The first part, $-\sum_{j\in\mathbb{N}_i}(q_{ij}^h - l_{ij}^h)$, referred to as the attractive force plays the role of forcing the agent i to its desired relative location with respect to the agent j defined by l_{ij}. On the other hand, the second part, $-\sum_{j\in\mathbb{N}_i}(a_{ij}\beta'_{ij}q_{ij}^h)$ referred to as the repulsive force, takes care of collision avoidance for the agent i with the other agents in the group when it is necessary. Interestingly, the second part can also be viewed as the gyroscopic force ([16]) to steer the agent i away from its group members when it is too close to them.

Remark 3: When $min(R_i, R_j) \geq \|l_{ij}\|$, the switching rule (7) should not be exactly performed at $\|q_{ij}\| = \|l_{ij}\|$ because it will be shown later that the distance between the agents i and j approaches $\|l_{ij}\|$ as the time tends to infinity with the help of the proposed control. Consequently, if the switching rule is performed at $\|q_{ij}\| = \|l_{ij}\|$ it will cause many switches in the switching parameter a_{ij} (so in the control u_i) when the controlled system is perturbed by an arbitrarily small noise. Therefore, when the switching rule (7) should be performed at $\|q_{ij}\| = \|l_{ij}\| + \varrho_{ij}$ or at $\|q_{ij}\| = \|l_{ij}\| - \varrho_{ij}$ with ϱ_{ij} a strictly positive constant and strictly smaller than $\|l_{ij}\|$.

Now substituting the control u_i given in (15) into (13) results in

$$\dot{\varphi}_i = \Omega_i^T\Psi_i(\Omega_i) - \sum_{j\in\mathbb{N}_i}(q_{ij} - l_{ij} + a_{ij}\beta'_{ij}q_{ij})^T(u_j - u_d) + \Phi_i(\eta - \eta_f) - (\eta - \eta_f)^T\Gamma(\eta - \eta_f). \tag{18}$$

On the other hand, substituting the control u_i given in (15) into (1) results in the closed loop system

$$\begin{aligned}
\dot{q}_i &= -C\Psi_i(\Omega_i) + u_d, i = 1, ..., N, \\
\dot{\eta} &= -\Gamma(\eta - \eta_f).
\end{aligned} \tag{19}$$

We now state the main result of our paper in the following theorem whose proof is given in the next section.

Theorem 1: Under the conditions specified in (2), the bounded controls $u_i, i = 1, ..., N$, and the adaptation rule for η given in (15) with the parameters a_{ij} chosen according to the switching rule (7) guarantee that no collisions between any agents can occur, the solutions of the closed loop system (19) exist, and the agents are globally asymptotically stabilized with respect to their group members in configurations that make a particular formation specified by the desired vector $\bar{l}(\eta_f) = [l_{12}^T(\eta_f), l_{13}^T(\eta_f), ..., l_{1N}^T(\eta_f), l_{23}^T(\eta_f), ..., l_{2N}^T(\eta_f), ..., l_{N-1,N}^T(\eta_f)]^T$, except for the set of measure zero defined in (2). The controls $u_i, i = 1, ..., N$, take the collision avoidance between the agents into account only when the agents are in their proximity. In addition, all the agents align their velocity vectors to the desired formation velocity vector u_d.

IV. Stability analysis: Proof of Theorem 1

Proof. For proof of Theorem 1, we use the following total potential function

$$\varphi = \frac{1}{2}\sum_{i=1}^{N}\varphi_i \tag{20}$$

whose derivative along the solutions of (18) is

$$\dot{\varphi} = -\sum_{i=1}^{N}\Omega_i^T C\Psi_i(\Omega_i) + \sum_{i=1}^{N}\Big(\Phi_i(\eta-\eta_f) - (\eta-\eta_f)^T\Gamma(\eta-\eta_f)\Big). \tag{21}$$

Since the switching parameter a_{ij} obeys the switching rule (7), the control u_i is a switching control. Hence the closed loop system (19) is a switched system. We will use the MLF approach to analyze stability of (19) based on (21). Let $t_k, k = 1, 2, ...$ be the switching times (i.e. when a_{ij} changes its value between 1 and 0, and vice versa). Let $\varphi_{\sigma_k}(t_{k_d})$ be the value of the function φ in the interval $[t_k, t_{k+1})$, i.e. $t_k \le t_{k_d} < t_{k+1}$. Since the function β_{ij} is zero at the switching times (i.e. when $\|q_{ij}\| = R_{ij}$), the values of the function $\varphi_{\sigma_k}(t_k)$ and $\varphi_{\sigma_{k-1}}(t_k)$ coincide at each switching time . This means that φ (we slightly abuse the notation, i.e. use φ instead of $\varphi_{\sigma_k}(t_k)$ for clarity) is a continuous Lyapunov function candidate. Therefore, using the stability results of a switched system based on the MLF approach (see [17], Chapter 3) we just need to investigate stability properties of the closed loop system (19) under a fixed $\mathbb{N}_i^*$, with $\mathbb{N}_i^*$ is the subset of $\mathbb{N}_i$ such that if $j \in \mathbb{N}_i^*$ then $a_{ij} = 1$. We first prove that no collisions between the agents can occur and that the agents are asymptotically stabilized at the desired or some critical configurations. Next, to investigate stability of the closed loop system (19) at these configurations, we linearize the closed loop system at these configuration. The direct Lyapunov method is then used to prove that only desired configuration is (unique) asymptotic stable and that other critical configurations are unstable.

+Proof of no collisions and existence of solutions
We first show that $\varphi(t)$ exists for all $t \ge t_0 \ge 0$ by considering the following potential function

$$\bar{\varphi} = \log(1+\varphi) + \frac{1}{2}\|\eta-\eta_f\|^2 \tag{22}$$

whose derivative along the solutions of the second equation of (19) and (21) satisfies

$$\dot{\bar{\varphi}} \le \frac{1}{1+\varphi}\sum_{i=1}^{N}\Big(\Phi_i(\eta-\eta_f)\Big) - \lambda_{min}(\Gamma)\|\eta-\eta_f\|^2 \tag{23}$$

where $\lambda_{min}(\Gamma)$ is the minimum eigenvalue of Γ. On the other hand, from Property 5) of β_{ij} and the expressions of Φ_i and φ, see (8), (12) and (20), there exist nonnegative constants ξ_1 and ξ_2 such that

$$\Big\|\sum_{i=1}^{N}\Phi_i\Big\| \le \xi_1\varphi + \xi_2. \tag{24}$$

Using (24), we can write (23) as

$$\dot{\bar{\varphi}} \leq \frac{(\xi_1 + \xi_2)^2}{4\lambda_{min}(\Gamma)} \tag{25}$$

which means that $\bar{\varphi}(t)$, so $\varphi(t)$, exists for all $t \geq t_0 \geq 0$. From the second equation of (19), we have

$$\|\eta(t) - \eta_f\| \leq \|\eta(t_0) - \eta_f\| e^{-\lambda_{min}(\Gamma)(t-t_0)} \tag{26}$$

which implies that the desired formation shape is globally exponentially achieved. Now using (26) and (24), we can write (21) as

$$\dot{\varphi} \leq (\xi_1 \varphi + \xi_2)\|\eta(t_0) - \eta_f\| e^{-\lambda_{min}(\Gamma)(t-t_0)} \tag{27}$$

which implies that

$$\varphi(t) \leq (\varphi(t_0) + \xi_2/\xi_1) e^{\xi_1 \|\eta(t_0) - \eta_f\|/\lambda_{min}(\Gamma)} := \varphi^M. \tag{28}$$

Substituting the expression of φ and φ_i given in (20) and (4) into (28) results in

$$0.5\Big(\gamma_i(t) + \beta_i(t) + 0.5\|\eta(t) - \eta_f\|\Big) \leq \Big(0.5(\gamma_i(t_0) + \beta_i(t_0) + 0.5\|\eta(t_0) - \eta_f\|) + \xi_2/\xi_1\Big) e^{\xi_1\|\eta(t_0)-\eta_f\|/\lambda_{min}(\Gamma)}. \tag{29}$$

Recalling from (6) that $\beta_i = \sum_{j\in\mathbb{N}_i}(a_{ij}\beta_{ij}) = \sum_{j\in\mathbb{N}_i^*}\beta_{ij}$. Therefore if $\mathbb{N}_i^*$ is empty, then there are no collisions since all $a_{ij} = 0$, i.e. $\|q_{ij}\| > R_{ij}$. On the other hand, if $\mathbb{N}_i^*$ is nonempty, from (2) and (8) we have $\beta_{ij}(t_0)$ is larger than some strictly positive constant. Therefore the right hand side of (29) is bounded by some positive constant depending on the initial conditions. Boundedness of the right hand side of (29) implies that the left hand side of (29) must be also bounded. As a result, $\beta_{ij}(t)$ must be larger than some strictly positive constant for all $t \geq t_0 \geq 0$. From properties of β_{ij}, see (8), $\|q_{ij}(t)\|$ must be larger than some strictly positive constant denoted by ϵ_4 , i.e. there are no collisions for all $t \geq t_0 \geq 0$. Boundedness of the left hand side of (29) also implies that of $\|q_{ij}(t)\|$ for all $t \geq t_0 \geq 0$. This readily implies that the solutions of the closed loop system (19) exist, since Ω_i is a function of q_{ij}, l_{ij} and u_d, see (14). Now using (28) and (24), we can write (21) as

$$\dot{\varphi} \leq -\sum_{i=1}^{N} \Omega_i^T C \Psi_i(\Omega_i) + (\xi_1 \varphi^M + \xi_2)\|\eta(t_0) - \eta_f\| e^{-\lambda_{min}(\Gamma)(t-t_0)} \tag{30}$$

where φ^M is defined in (28). Applying Barbalat's lemma found in [18] to (30) yields

$$\lim_{t\to\infty} \Omega_i^T(t) C \Psi_i(\Omega_i(t)) = 0, \forall i = 1, 2, ..., N. \tag{31}$$

Thanks to Property 2) of the function ψ_i, see (17), the limit equation (31) implies that

$$\lim_{t\to\infty} \Omega_i(t) = \lim_{t\to\infty} \Big[\sum_{j\in\mathbb{N}_i}(q_{ij}(t) - l_{ij}) + \sum_{j\in\mathbb{N}_i^*} \beta'_{ij}(t) q_{ij}(t)\Big] = 0, \forall i = 1, 2, ..., N. \tag{32}$$

The limit equation (32) implies that $q_{ij}(t)$ tends to $l_{ij}(\eta_f)$ or some constant vector q_{ijc} as the time goes to infinity. The constant vector q_{ijc} is such that

$$\sum_{j\in\mathbb{N}_i}(q_{ijc} - l_{ij}) + \sum_{j\in\mathbb{N}_i^*} \beta'_{ijc} q_{ijc} = 0. \tag{33}$$

Next, we will show that the desired configuration specified by $\bar{l}(\eta_f) = [l_{12}^T(\eta_f), l_{13}^T(\eta_f), ..., l_{1N}^T(\eta_f), l_{23}^T(\eta_f), l_{24}^T(\eta_f), ..., l_{2N}^T(\eta_f), ..., l_{N-1,N}^T(\eta_f)]^T$ is asymptotically stable, and that the critical configuration specified by $\bar{q}_c = [q_{12c}^T, q_{13c}^T, ..., q_{1Nc}^T, q_{23c}^T, q_{24c}^T, ..., q_{2Nc}^T, ..., q_{N-1,Nc}^T]^T$ is unstable by linearizing the closed loop system (19) at these configurations. Since we are investigating properties of the closed loop system (19) at the aforementioned configurations as the time tends to infinity, it is sufficient to assume that the formation parameter η is already equal to η_f. Moreover, since the aforementioned configurations are specified in terms of relative distances between the agents, it is much convenient to look at the inter-agent closed loop system derived from the closed loop system (19) as

$$\dot{\bar{q}} = -\bar{C}\bar{F}(\bar{q}) \tag{34}$$

where $\bar{q} = [q_{12}^T, q_{13}^T, ..., q_{1N}^T, q_{23}^T, q_{24}^T, ..., q_{2N}^T, ..., q_{N-1,N}^T]^T$, $\bar{C} = \text{diag}(C, ..., C)$, and $\bar{F}(\bar{q}) = \big[\Psi_1^T(\Omega_1) - \Psi_2^T(\Omega_2), \Psi_1^T(\Omega_1) - \Psi_3^T(\Omega_3), ..., \Psi_1^T(\Omega_1) - \Psi_N^T(\Omega_N), \Psi_2^T(\Omega_2) - \Psi_3^T(\Omega_3), ..., \Psi_2^T(\Omega_2) - \Psi_N^T(\Omega_N), ..., \Psi_{N-1}^T(\Omega_{N-1}) - \Psi_N^T(\Omega_N)\big]^T$. Linearizing the inter-agent closed loop system (34) at $\bar{q}_o$, which can be either $\bar{l}(\eta_f)$ or $\bar{q}_c$ results in

$$\dot{\bar{q}} = -\bar{C}\partial\bar{F}(\bar{q})/\partial\bar{q}\big|_{\bar{q}=\bar{q}_o}(\bar{q} - \bar{q}_o) \tag{35}$$

where

$$\frac{\partial\bar{F}(\bar{q})}{\partial\bar{q}} = \begin{bmatrix} \Delta_{12}^{12} & \Delta_{12}^{13} & \cdots & \cdots & \Delta_{12}^{N-1,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{ij}^{12} & \cdots & \Delta_{ij}^{ij} & \cdots & \Delta_{ij}^{N-1,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta_{N-1,N}^{12} & \cdots & \cdots & \cdots & \Delta_{N-1,N}^{N-1,N} \end{bmatrix} \tag{36}$$

with $\Delta_{ij}^{hk} = \frac{\partial\Psi_i(\Omega_i)}{\partial\Omega_i}\frac{\partial\Omega_i}{\partial q_{hk}} - \frac{\partial\Psi_j(\Omega_j)}{\partial\Omega_j}\frac{\partial\Omega_j}{\partial q_{hk}}, (i,j) \in (1, ..., N), i \neq j$, and $(h,k) \in (1, ..., N), h \neq k$. We now investigate properties of $\bar{l}(\eta_f)$ and $\bar{q}_c$ based on (36).

- *Proof of $\bar{l}(\eta_f)$ being asymptotic stable:* Consider the following Lyapunov function candidate

$$V_{\bar{l}} = \frac{1}{2}(\bar{q} - \bar{l})^T(\bar{q} - \bar{l}) \tag{37}$$

whose derivative along the solutions of (35) with $\bar{q}_o$ replaced by $\bar{l}(\eta_f)$ satisfies

$$\dot{V}_{\bar{l}} = -\frac{N(N-1)}{2}(\bar{q} - \bar{l})^T(\bar{q} - \bar{l}) - \frac{N(N-1)}{4}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i^*}\beta_{ijl}''\big(l_{ij}^T(q_{ij} - l_{ij})\big)^2 \tag{38}$$

where $\beta_{ijl}'' = \beta_{ij}''\big|_{\|q_{ij}\|=\|l_{ij}\|}$, and we have used Property 4) of β_{ij}, see (8), i.e. $\beta_{ijl}' = 0$, with $\beta_{ijl}' = \beta_{ij}'\big|_{\|q_{ij}\|=\|l_{ij}\|}$. Furthermore, from Property 4) of β_{ij}, see (8), we have $\beta_{ijl}'' \geq 0$. Substituting $\beta_{ijl}'' \geq 0$ into (38) gives

$$\dot{V}_{\bar{l}} \leq -\frac{N(N-1)}{2}(\bar{q} - \bar{l})^T(\bar{q} - \bar{l}) \tag{39}$$

which together with (37) imply that $\bar{l}(\eta_f)$ is asymptotically stable.

- *Proof of $\bar{q}_c$ being unstable:* Consider the following Lyapunov function candidate

$$V_{\bar{q}_c} = \frac{1}{2}(\bar{q} - \bar{q}_c)^T(\bar{q} - \bar{q}_c) \tag{40}$$

whose derivative along the solutions of (35) with $\bar{q}_o$ replaced by $\bar{q}_c$ satisfies

$$\begin{aligned}\dot{V}_{\bar{q}_c} &= -\frac{N(N-1)}{4}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}(q_{ij}-q_{ijc})^T\big(I_{n\times n}+a_{ij}\beta'_{ijc}I_{n\times n}+a_{ij}\beta''_{ijc}q_{ijc}q_{ijc}^T\big)(q_{ij}-q_{ijc}),\\ &= -\frac{N(N-1)}{4}\sum_{i=1}^{N}\Big[\sum_{j\in\mathbb{N}_i}(q_{ij}-q_{ijc})^T\big(I_{n\times n}+a_{ij}\beta'_{ijc}I_{n\times n}\big)(q_{ij}-q_{ijc})+\\ &\quad \sum_{j\in\mathbb{N}_i^*}\beta''_{ijc}(q_{ijc}^T(q_{ij}-q_{ijc}))^2\Big].\end{aligned} \tag{41}$$

Now defining $\bar{\Omega}=\big[\Omega_1^T-\Omega_2^T,\Omega_1^T-\Omega_3^T,...,\Omega_1^T-\Omega_N^T,\Omega_2^T-\Omega_3^T,...,\Omega_2^T-\Omega_N^T,...,\Omega_{N-1}^T-\Omega_N^T\big]^T$, we have $\bar{\Omega}_c=\bar{\Omega}\big|_{\bar{q}=\bar{q}_c}=0$. Multiplying both sides of $\bar{\Omega}_c=0$ with $\bar{q}_c^T$ results in $\bar{q}_c^T\bar{\Omega}_c=0$. From the expression of $\bar{\Omega}$ with $\bar{q}$ replaced by $\bar{q}_c$, we expand $\bar{q}_c^T\bar{\Omega}_c=0$ to

$$\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\big[q_{ijc}^T(q_{ijc}-l_{ij})+a_{ij}\beta'_{ijc}q_{ijc}^Tq_{ijc}\big]=0\Longrightarrow\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\big[(1+a_{ij}\beta'_{ijc})q_{ijc}^Tq_{ijc}\big]=\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}(q_{ijc}^Tl_{ij}). \tag{42}$$

The sum $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}(q_{ijc}^Tl_{ij})$ is strictly negative since at the point F where $q_{ij}=l_{ij},\ \forall(i,j)\in\{1,...,N\},i\neq j$ all attractive and repulsive forces are equal to zero while at the point C where $q_{ij}=q_{ijc}\ \forall(i,j)\in\{1,...,N\},i\neq j$ the sum of attractive and repulsive forces are equal to zero (but attractive and repulsive forces are nonzero). Therefore the point O where $q_{ij}=0,\ \forall(i,j)\in\{1,...,N\},i\neq j$ must locate between the points F and C for all $(i,j)\in\{1,...,N\},i\neq j$. That is there exists a strictly positive constant b such that $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}(q_{ijc}^Tl_{ij})\leq -b$. Substituting $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}(q_{ijc}^Tl_{ij})\leq -b$ into (42) gives

$$\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\big[(1+a_{ij}\beta'_{ijc})q_{ijc}^Tq_{ijc}\big]\leq -b \tag{43}$$

which implies that there must be at least one pair $(i^*,j^*)\in\{1,...,N\},i^*\neq j^*$ such that

$$1+a_{i^*j^*}\beta'_{i^*j^*c}\leq -b^* \tag{44}$$

where b^* is a strictly positive constant, and indeed $a_{i^*j^*}=1$. We now write (41) as

$$\begin{aligned}\dot{V}_{\bar{q}_c}=-\frac{N(N-1)}{4}\Big[&(q_{i^*j^*}-q_{i^*j^*c})^T(1+a_{i^*j^*}\beta'_{i^*j^*c})(q_{i^*j^*}-q_{i^*j^*c})+\\ &\sum_{i=1,i\neq i^*}^{N}\ \sum_{j\in\mathbb{N}_i,j\neq j^*}(q_{ij}-q_{ijc})^T\big(I_{n\times n}+a_{ij}\beta'_{ijc}I_{n\times n}\big)(q_{ij}-q_{ijc})+\\ &\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i^*}\beta''_{ijc}(q_{ijc}^T(q_{ij}-q_{ijc}))^2\Big].\end{aligned} \tag{45}$$

Define a subspace such that in this subspace $q_{ij}=q_{ijc},\forall(i,j)\in\{1,...,N\},i\neq i^*,j\neq j^*,i\neq j$ and $q_{ij}^T(q_{ij}-q_{ijc})=0,\ \forall(i,j)\in\{1,...,N\},i\neq j$. Therefore in this subspace, we have from (40) and (45) that

$$\begin{aligned}V_{\bar{q}_c} &= \frac{1}{2}(q_{i^*j^*}-q_{i^*j^*c})^T(q_{i^*j^*}-q_{i^*j^*c}),\\ \dot{V}_{\bar{q}_c} &= -\frac{N(N-1)}{4}(q_{i^*j^*}-q_{i^*j^*c})^T(1+a_{i^*j^*}\beta'_{i^*j^*c})(q_{i^*j^*}-q_{i^*j^*c}),\\ &\geq \frac{b^*N(N-1)}{4}(q_{i^*j^*}-q_{i^*j^*c})^T(q_{i^*j^*}-q_{i^*j^*c})\end{aligned} \tag{46}$$

where we have used (44). The set of equations in (46) clearly imply that $\bar{q}_c$ is unstable. Proof of Theorem 1 is completed.

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